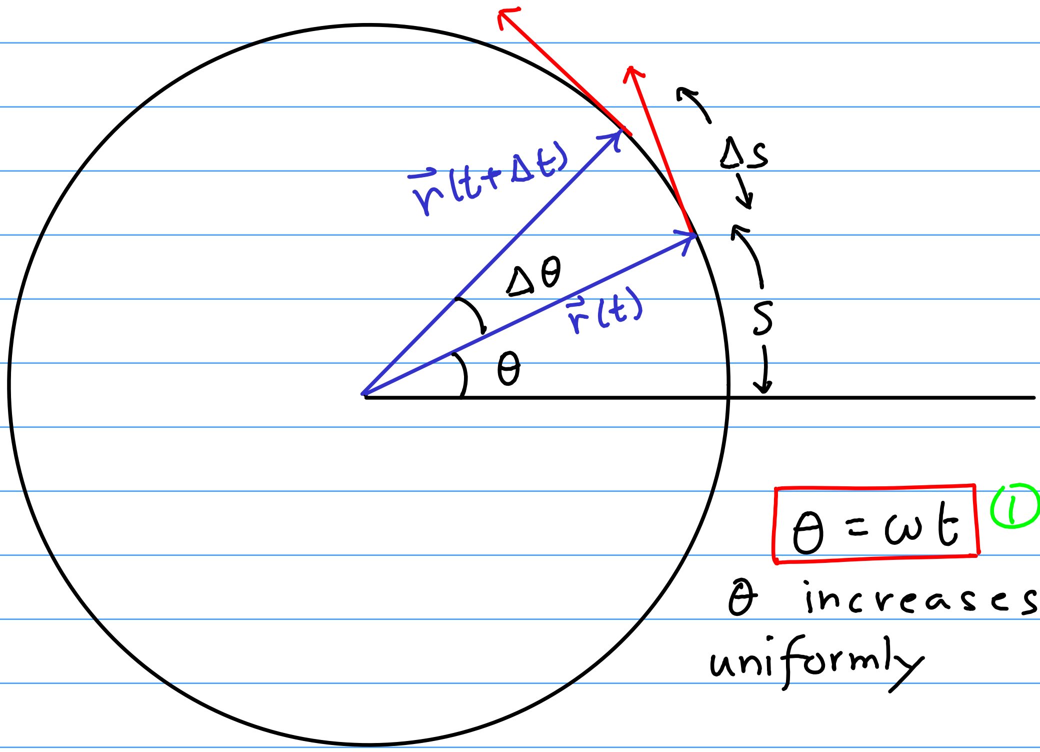
Circular Motion

First consider uniform circular motion, where the body moves at constant speed in a circle



$$\boxed{\theta = \omega t} \quad (1)$$

θ increases uniformly

First the fast calculus way

$$\boxed{\vec{r}(t) = R\left(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right)} \quad (2)$$

$$\boxed{\vec{v}(t) = \frac{d\vec{r}}{dt} = R\omega\left(-\sin(\omega t)\,\hat{i} + \cos\omega t\,\hat{j}\right)} \quad (3)$$

$$\boxed{\vec{a}(t) = \frac{d\vec{v}}{dt} = -R\omega^2\left(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right)} \quad (4)$$

$$\boxed{\vec{a}(t) = -\omega^2\,\vec{r}(t)} \quad (5)$$

What is the connection of ω to the speed?

$\theta = \omega t$ $\qquad$ $\boxed{s = R\theta = R\omega t}$ (6)

and $\boxed{v = \frac{ds}{dt} = R\omega}$ (7)

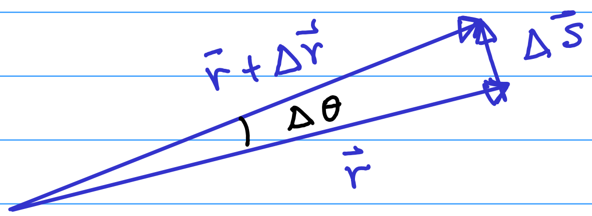
So $\boxed{\omega = \text{angular velocity} = \frac{v}{R}}$ (8)

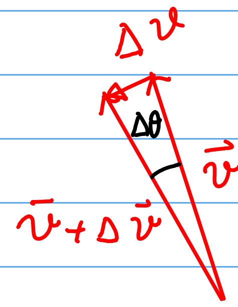
So $\boxed{\vec{a} = -\omega^2 \vec{r}(t) = -\frac{v^2}{R}\left(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right)}$ (9)

So the magnitude of $\vec{a}$ is $\boxed{\frac{v^2}{R} = a_c}$ (10)

a_c = Centripetal acceleration and the direction is towards the center of the circle.

Now the geometric method





By similar triangles

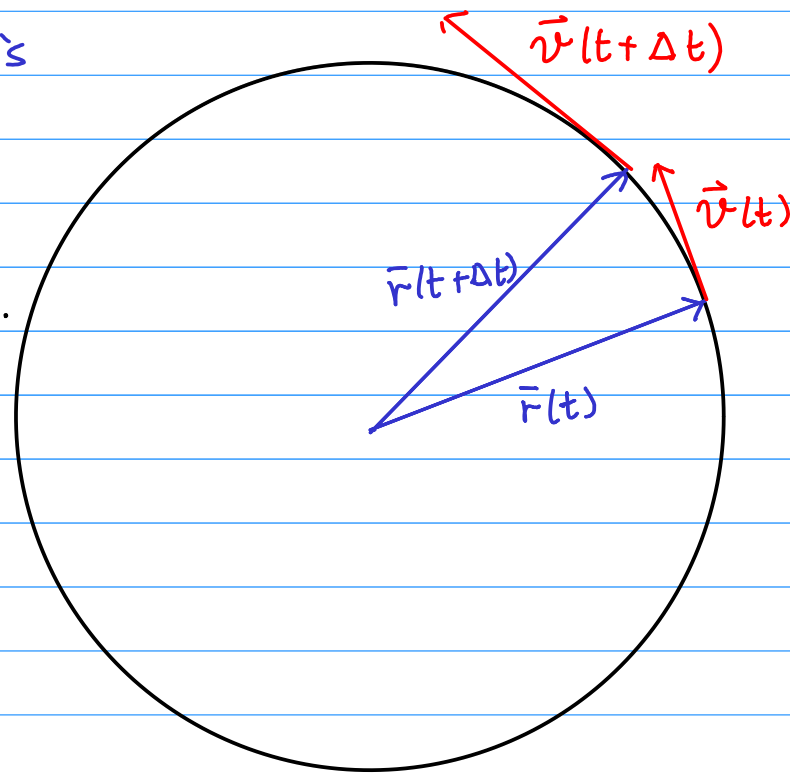
$\boxed{\frac{\Delta s}{R} = \frac{\Delta v}{v}}$ (11) $\Rightarrow$ $\boxed{\frac{1}{R}\frac{\Delta s}{\Delta t} = \frac{1}{v}\frac{\Delta v}{\Delta t}}$ (12)

or taking $\Delta t \to 0$ $\quad \frac{\Delta s}{\Delta t} = v \quad \frac{\Delta v}{\Delta t} = |\vec{a}_c|$

$$\frac{v}{R} = \frac{a_c}{v} \quad \text{or} \quad \boxed{a_c = \frac{v^2}{R}} \quad (13)$$

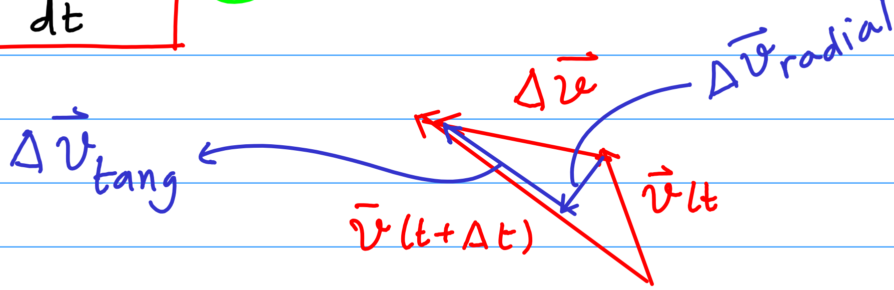
Now this is for uniform circular motion. Suppose we allow the speed to change as well. Then there will be two components to the acceleration.

The radial part is still $\vec{a}_c = -\frac{v^2}{R^2}\,\vec{r}(t)$ because the velocity is changing direction.



However, now there is also a tangential part to the acceleration

$$\boxed{a_t = \frac{dv}{dt}} \quad (14)$$

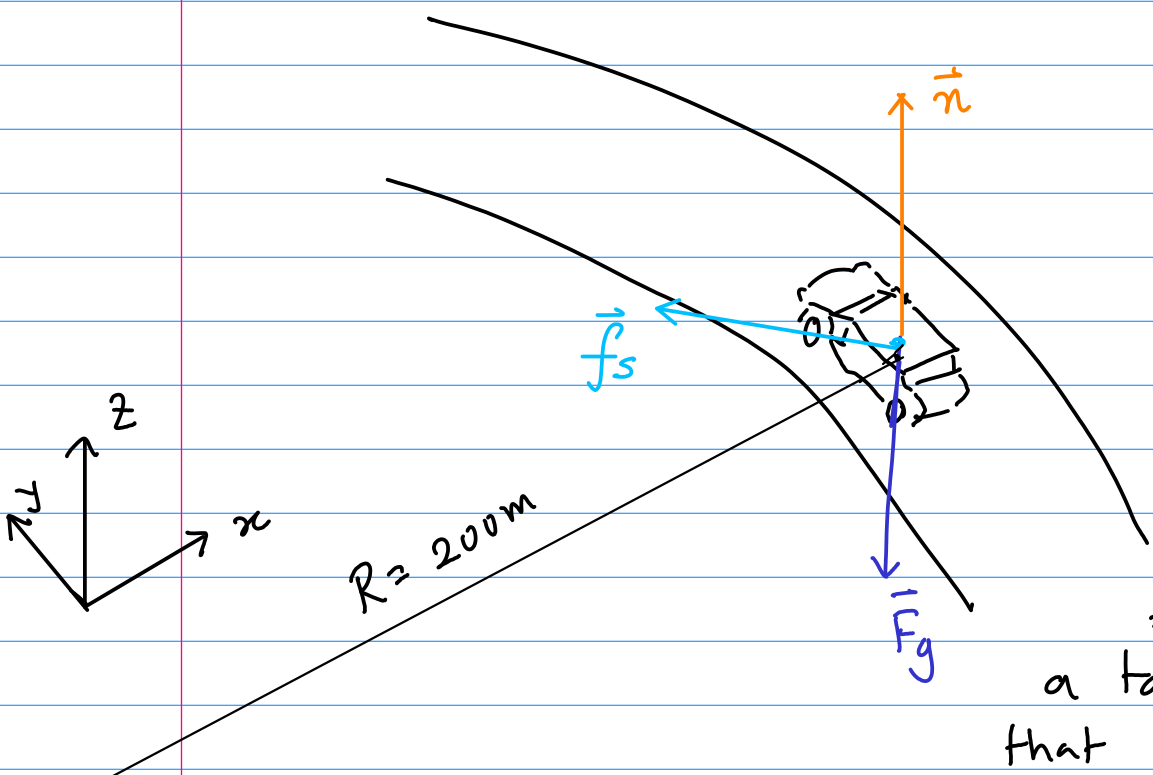


As a vector $\boxed{\vec{a}_t = \frac{1}{v}\frac{dv}{dt}\,\vec{v}} \quad (15)$

Always remember that one needs to add the two components like vectors to get the total acceleration.

Example: A car starts from rest on a horizontal circular track and accelerates at a constant rate of $\boxed{a_t = 3\,m/s^2}$ 16. The coefficient of static friction between the tires and the track is $\boxed{\mu_s = 0.5}$ 17 and the radius of the track is $\boxed{200\,m.}$ 18 When does the car skid?

We need to draw the FBD of the car from two different perspectives. First looking at it from an angle



$\vec{f_s}$ = Force of static friction. This has a radial part which accelerates the car towards the center and a tangential part that accelerates the car along the track.

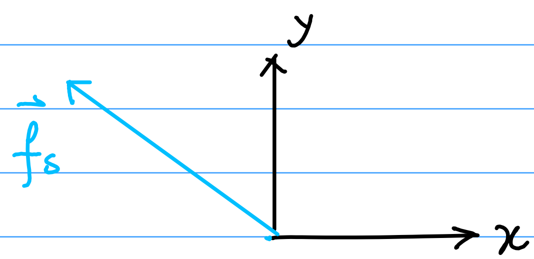
We need to use all 3 dimensions for this

$$\boxed{\vec{n} = n\hat{k}} \quad (19) \qquad (\hat{k} = \text{unit vector in the } z \text{ direction})$$

$$\boxed{\vec{F}_g = -mg\,\hat{k}} \quad (20)$$

Choose the coordinates shown at the bottom right. Then, in the xy plane we have

$\vec{f}_s$ being



The x-component of $\vec{f}_s$ is the radial force and the y-component the tangential force

$$\boxed{\vec{f}_s = -f_r\,\hat{i} + f_t\,\hat{j}} \quad (21)$$

$$\boxed{\vec{F}_{tot} = -f_r\,\hat{i} + f_t\,\hat{j} + (n - mg)\,\hat{k}} \quad (22)$$

We know that $\boxed{a_z = 0}$ (23) $\Rightarrow$ $\boxed{F_{tot,z} = 0}$ (24)

$$\Rightarrow \boxed{n = mg} \quad (25)$$

So $$\boxed{-f_r\,\hat{i} + f_t\,\hat{j} = m\,\vec{a}} \quad (26)$$

$\vec{a}$ has a radial part (in the $-x$ direction) which is centripetal, and a tangential part which is $a_t = 3\,m/s^2$

$$\boxed{\vec{a} = -\frac{v^2}{R}\hat{i} + 3\frac{m}{s^2}\hat{j}}$$ 27

So. $$-f_r\hat{i} + f_t\hat{j} = -\frac{mv^2}{R}\hat{i} + m\left(3\frac{m}{s^2}\right)\hat{j}$$

magnitude of $\vec{f}$ is $\boxed{f = \sqrt{f_r^2 + f_t^2}}$ 28

The maximum force of static friction is

$$\boxed{f_{s,max} = \mu_s n = \mu_s mg = m|\vec{a}|}$$ 29

$$\Rightarrow \boxed{|\vec{a}|_{max} = \mu_s g = 4.9\,m/s^2}$$ 30

$$\boxed{|\vec{a}| = \sqrt{\left(\frac{v^2}{R}\right)^2 + a_t^2}}$$ 31

32

Since the car starts from rest $\boxed{v(t) = a_t t}$

$$\boxed{|\vec{a}| = \sqrt{\frac{a_t^4 t^4}{R^2} + a_t^2}}$$ 33

$\Rightarrow$ Just on the verge of skidding

$\boxed{|\vec{a}| = |\vec{a}|_{max}}$ 34 $\Rightarrow \frac{a_t^4 t^4}{R^2} + a_t^2 = (\mu_s g)^2$

$$\frac{a_t^2 t^2}{R} = \sqrt{(\mu_s g)^2 - a_t^2} = \left[(4.9)^2 - 3^2\right]^{\frac{1}{2}} = 3.87\,m/s^2$$

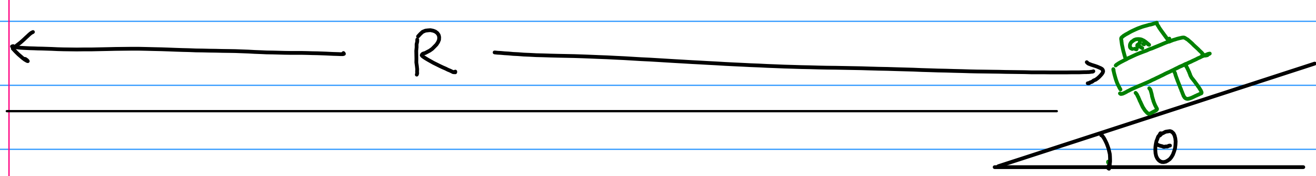
$\Rightarrow \quad t^2 = \dfrac{200\,m \cdot 3.87\,m/s^2}{(3\,m/s^2)^2} = 86.09\,s^2$

$\boxed{t = 9.28\ sec}$ 35

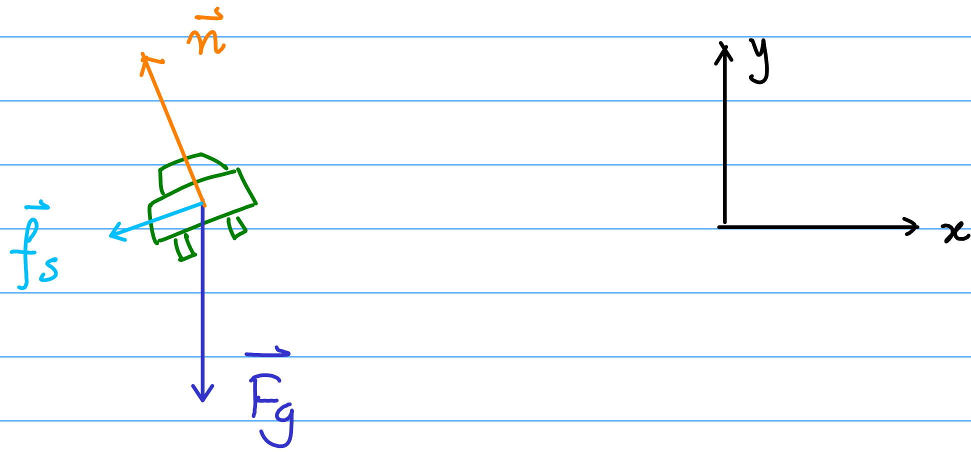
The car's speed at this time is

$\boxed{v_{skid} = 3\dfrac{m}{s^2} \times 9.28\ sec = 27.84\ m/s}$ 36

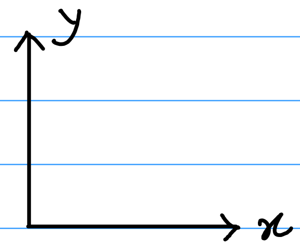
Banked curve: Normally, on a highway, the curves are banked, which means they are not horizontal, but form a slope tilted inward towards the center. Consider a car travelling at constant speed v on such a curve of radius R and banking angle θ as shown. The car is going into the page at the moment.



FBD



What coordinates should we choose? Here the most convenient coordinates are



because we know that $a_y = 0$ while

$$\boxed{a_x = -\frac{v^2}{R}} \quad (37)$$

(uniform circular motion)

Break forces into components

$$\boxed{\vec{F}_g = -mg\,\hat{j}} \quad (38) \qquad \boxed{\vec{n} = -n\sin\theta\,\hat{i} + n\cos\theta\,\hat{j}} \quad (39)$$

$$\boxed{\vec{f}_s = -f_s\cos\theta\,\hat{i} - f_s\sin\theta\,\hat{j}} \quad (40)$$

$$\boxed{\vec{F}_{tot} = (-n\sin\theta - f_s\cos\theta)\,\hat{i} + (n\cos\theta - f_s\sin\theta - mg)\,\hat{j}} \quad (41)$$

Since the car is not accelerating vertically $\boxed{a_y = 0}$ (42)

$$\Rightarrow \boxed{n\cos\theta - f_s\sin\theta = mg} \quad (43)$$

Since $a_x = -\frac{v^2}{R}$

$$\boxed{-n\sin\theta - f_s\cos\theta = -\frac{mv^2}{R}} \quad (44)$$

45

Let's put in some numbers. Say $\boxed{R = 250\,m}$

46 47

$\boxed{v = 25\,m/s}$ $\boxed{\theta = 15^\circ = \frac{\pi}{12}\text{ rad}}$ and $\boxed{\mu_s = 0.5}$ 48

Will the car skid? Let us see what the required f_s is. If this is greater than

$f_{s,max} = \mu_s n$ then the car skids.

49

$\boxed{\sin\theta = 0.259}$ $\boxed{\cos\theta = 0.966}$ 50

Let $\boxed{m = 2000\,kg}$ 51

52

$$\boxed{\begin{aligned} n\,(0.966) - f_s\,(0.259) &= 2000 \times 9.8 = 19600\ N \\ n\,(0.259) + f_s\,(0.966) &= 2000 \times \frac{(25)^2}{250} = 5000\ N \end{aligned}}$$

Solve the simultaneous eq^{n}s to get

$\boxed{n = 20229\ N}$ 53 $\boxed{f_s = -246.4\ N}$ 54

So we guessed wrong about the direction of f_s but the answer is algebraically correct.

55

Note that $\boxed{n > mg}$ here! Because n is providing part of the force needed to keep the car moving in a circle.

$$\boxed{|f_{s,max}| = \mu_s n = 10114\ N > |-246.4\ N|}$$ 56

$\Rightarrow$ the car does not skid.

In fact, the car is going very close to the rated speed of the curve, the speed at which friction is not needed to keep it on the curve.

To find the rated speed v_0 set $f_s = 0$

$$\boxed{n \cos\theta = mg} \quad 57 \quad \Rightarrow \quad \boxed{n = \frac{mg}{\cos\theta}} \quad 58$$

$$\boxed{n \sin\theta = \frac{mv^2}{R}} \quad 59 \quad \Rightarrow \quad \boxed{mg \tan\theta = mv^2/R} \quad 60$$

$$\Rightarrow \boxed{v_0 = \sqrt{gR\tan\theta} = \sqrt{9.8\, m/s^2 \; 250\, m \; \tan \pi/12} = 25.6\, m/s^2} \quad 61$$

For $v < v_0$, the force of friction points outward along the surface, while for $v > v_0$ f_s points inward.

<u>Example 2</u>: How fast does the car have to be going for it to be on the verge of a skid?

If it is on the verge of skidding

$$\boxed{|f_s| = f_{s,max} = \mu_s n} \quad 62$$

Go back to

$$\boxed{\begin{aligned} n\cos\theta - f_s\sin\theta &= mg \\ n\sin\theta + f_s\cos\theta &= \frac{mv_{max}^2}{R} \end{aligned}}$$ (63)

Now $f_s = \mu_s n$

$\Rightarrow$ $$\boxed{n(\cos\theta - \mu_s\sin\theta) = mg}$$ (64)

or $$\boxed{n = \frac{mg}{\cos\theta - \mu_s\sin\theta} = \frac{19600\,N}{0.966 - 0.5\times 0.259} = 23428\,N}$$ (65)

Now $$\boxed{n\sin\theta + f_s\cos\theta = n(\sin\theta + \mu_s\cos\theta) = 23428(0.259 + 0.5\times 0.966) = 17384\,N}$$ (66)

But this must be mv_{max}^2/R

$\Rightarrow$ $$\boxed{\frac{v_{max}^2}{R} = \frac{17384}{2000} = 8.69\ m/s^2}$$ (67)

$\Rightarrow$ $v^2 = 200\,m \times 8.69\,m/s^2 = 1738\,(m/s)^2$

$$\boxed{v_{max} = 41.7\ m/s}$$ (68)