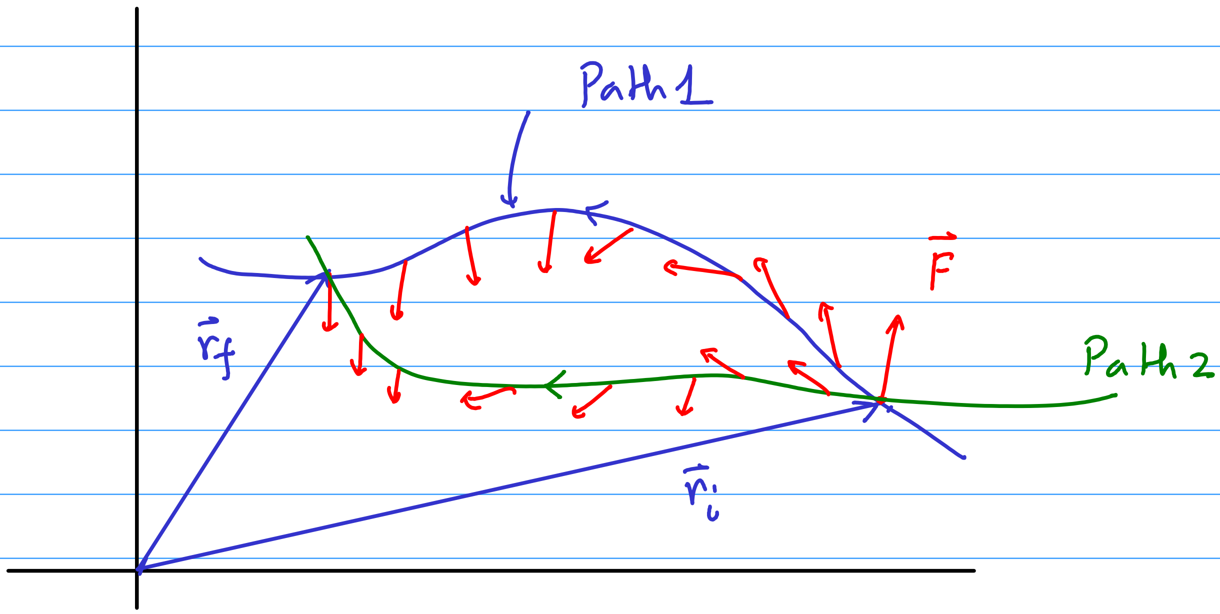
Conservative / Nonconservative forces and Potential Energy

For a force varying with position the work done along a path from $\vec{r}_i$ to $\vec{r}_f$ is



A conservative force is one for which

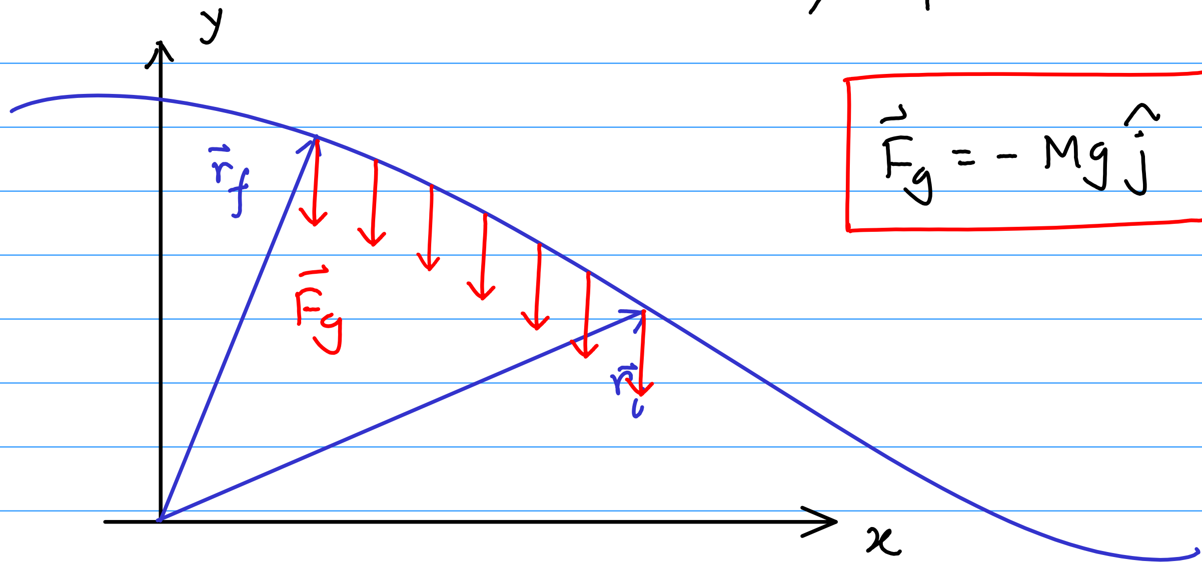
the work done depends only on $\vec{r}_i$ and $\vec{r}_f$ and not on the path.

So the answer should be the same for the blue path and the green path and in fact any other path

A force for which the work done does depend on the path is nonconservative.

Friction of any kind is nonconservative. This is clear for kinetic friction because the longer the distance the more the (negative) work done by kinetic friction. One can take paths of many lengths between $\vec{r}_i$ and $\vec{r}_f$ and the W_f will definitely depend on the path.

Now for some conservative forces: $\vec{F}_g$ is conservative. Think about any path



$$\boxed{\vec{F}_g = -Mg\,\hat{j}} \quad (1)$$

In any tiny segment of the path

$$\boxed{d\vec{r} = dx\,\hat{i} + dy\,\hat{j}} \quad (2) \qquad \boxed{\vec{F}_g \cdot d\vec{r} = -Mg\,dy} \quad (3)$$

So all that matters is the change in y over the path

$$\boxed{\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = -Mg \int_{\vec{r}_i}^{\vec{r}_f} dy = -Mg\,y_f + Mg\,y_i} \quad (4)$$

The spring restoring force is conservative.

$$\boxed{\int_{\Delta x_i}^{\Delta x_f} -kx' \, dx' = -\frac{1}{2} k (\Delta x_f)^2 + \frac{1}{2} k (\Delta x_i)^2} \quad (5)$$

where Δx_i is the initial extention of the spring and Δx_f is the final extention.

So, what is the big deal with conservative forces? Why do we make the distinction between conservative and nonconservative?

The work done by a conservative force can be expressed as a change in Potential Energy U.

Let us define the gravitational PE U_g as

$$\boxed{U_g(\vec{r}) = Mgy} \quad (6)$$

Then

$$\boxed{\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_g \cdot d\vec{r} = -U_g(\vec{r}_f) + U_g(\vec{r}_i)} \quad (7)$$

Similarly

$$\boxed{\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_s \cdot d\vec{r} = -U_s(\vec{r}_f) + U_s(\vec{r}_i)} \quad (8)$$

To see why conservative forces are special, let us consider a particle acted upon only by conservative forces.

So,
$$\boxed{\int_{\bar{r}_i}^{\bar{r}_f} \bar{F}_{tot} \cdot d\bar{r} = -U(\bar{r}_f) + U(\bar{r}_i)} \quad (9)$$

But, by the Work-KE Thm
$$\boxed{\int_{\bar{r}_i}^{\bar{r}_f} \bar{F}_{tot} \cdot d\bar{r} = \Delta K = K_f - K_i} \quad (10)$$

$$\Rightarrow \quad \boxed{K_f - K_i = -U(\bar{r}_f) + U(\bar{r}_i)} \quad (11)$$

$$\Rightarrow \quad \boxed{K_f + U(\bar{r}_f) = K_i + U(\bar{r}_i)} \quad (12)$$

$$\boxed{\frac{1}{2} M \vec{v}_f^{\,2} + U(\bar{r}_f) = \frac{1}{2} M \vec{v}_i^{\,2} + U(\bar{r}_i)} \quad (13)$$

This is the 1st example of a conservation law. It says that the sum of the kinetic and potential energies is conserved (constant)

This sum is the mechanical energy
$$E_{mech} = K + U$$

Of course, this holds only if all forces that do work are conservative.

What happens when some forces acting on the body are conservative ($\bar{F}_c$) and others ($\bar{F}_{nc}$) are not?

$$\boxed{\vec{F}_{tot} = \vec{F}_c + \vec{F}_{nc}} \quad (14)$$

$$\boxed{W_{tot} = \int_{\bar{r}_i}^{\bar{r}_f} \vec{F}_{tot} \cdot d\bar{r} = \int_{\bar{r}_i}^{\bar{r}_f} \vec{F}_c \cdot d\bar{r} + W_{nc}} \quad (15)$$

$$\boxed{W_{tot} = -U(\bar{r}_f) + U(\bar{r}_i) + W_{nc}} \quad (16)$$

By the Work-KE Thm

$$\boxed{W_{tot} = K_f - K_i} \quad (17)$$

So

$$K_f - K_i = -U(\bar{r}_f) + U(\bar{r}_i) + W_{nc}$$

or

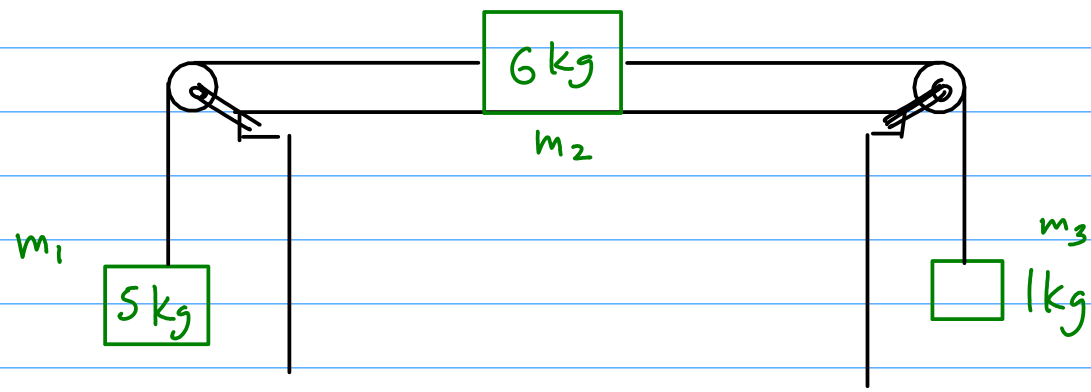
$$\boxed{K_f + U(\bar{r}_f) = K_i + U(\bar{r}_i) + W_{nc}} \quad (18)$$

Recall $$\boxed{E_{mech} = K + U} \quad (19)$$

$$\Rightarrow \boxed{E_{mech,f} = E_{mech,i} + W_{nc}} \quad (20)$$

or $$\boxed{W_{nc} = \Delta E_{mech}} \quad (21)$$

Example 1:



The 6 kg mass starts at rest and moves 1m to the left. Its final velocity is 1 m/s. What is μ_k?

We need to use $\boxed{W_{nc} = E_{mech,f} - E_{mech,i}}$ (22)

Initially $\boxed{v_{i,1} = v_{i,2} = v_{i,3} = 0 \quad = K_i = 0}$ (23)

Finally $\boxed{v_{f1} = v_{f2} = v_{f3} = 1 \text{ m/s}}$ (24)

$\Rightarrow$ $\boxed{K_f = \frac{1}{2}(5kg + 6kg + 1kg)(1 \text{ m/s})^2}$ (25)

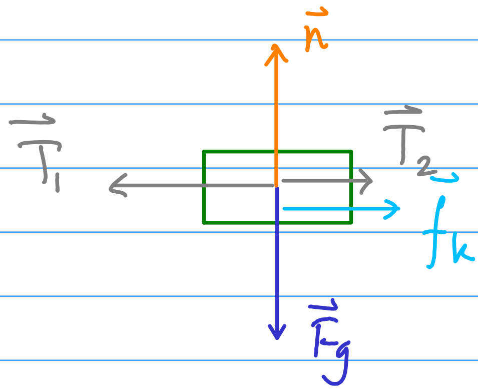
$\boxed{= 6 \text{ J}}$ (26)

How about U? The 6kg mass remains at constant y and does not change U_g.

The 5kg mass moves down 1m while the 1kg mass moves up 1m

$$\boxed{U_f - U_i = -5\,kg \times 9.8\,m/s^2 \times 1\,m + 1\,kg \times 9.8\,m/s^2 \times 1\,m = -39.2\,J}$$ (27)

Now we need to draw the FBD for the 6kg mass



It is clear that since $a_y = 0$ $\boxed{n = m_2 g}$ (28)

So $\boxed{f_k = \mu_k m_2 g}$ (29)

(30)

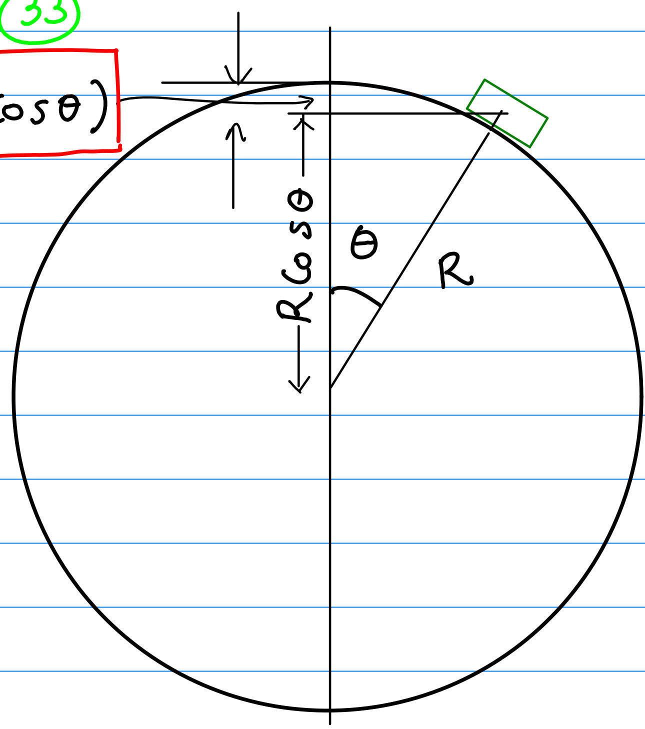
$$\boxed{\begin{aligned} W_{nc} = -f_k \Delta x &= E_{mech,f} - E_{mech,i} \\ &= K_f - K_i + U_f - U_i \\ &= 6J - 39.2\,J = -33.2\,J \end{aligned}}$$

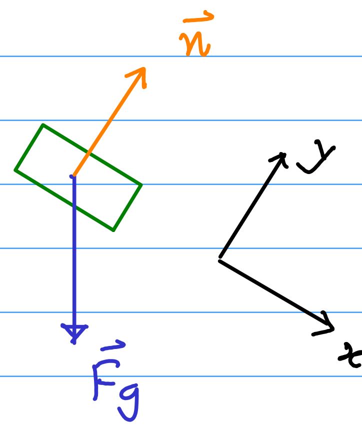
$\Rightarrow$ $\boxed{-\mu_k \times 6\,kg \times 9.8\,m/s^2 \times 1\,m = -33.2\,J}$ (31)

$\boxed{\mu_k = 0.564}$ (32)

Example 2: A block of negligible size sits on the top of a frictionless sphere of radius 0.5m. An extremely tiny nudge is given so the block starts sliding to the right. At what angle will it lose contact with the sphere?

$$\boxed{h_f = R(1-\cos\theta)} \quad (33)$$





The block has both $\vec{a}_r$ and $\vec{a}_t$ because it speeds up as it slides down.

Let us assume it is still in contact with the sphere at θ. Choosing the initial position as height $h_i = 0$

$$\boxed{E_{mech,i} = K_i + U_i = \frac{1}{2}Mv_i^2 + Mgh_i = 0} \quad (34)$$

$$\boxed{E_{mech}(\theta) = \frac{1}{2}M\left[v(\theta)\right]^2 - MgR(1-\cos\theta)} \quad (35)$$

Since no work is done by nonconservative forces E_{mech} is conserved.

$$\Rightarrow \quad \boxed{E_{mech}(\theta) = E_{mech,i} = 0} \quad (36)$$

$$\Rightarrow \quad \boxed{v(\theta) = \sqrt{2gR(1-\cos\theta)}} \quad (37)$$

This tells us that the centripetal acc is

$$\boxed{a_c(\theta) = \frac{v^2(\theta)}{R} = 2g(1-\cos\theta)} \quad (38)$$

Choose the coordinate system where x is tangent to the sphere. Then

$$\boxed{\vec{n} = n\hat{j}} \quad (39) \qquad \boxed{\vec{F}_g = Mg\sin\theta\,\hat{i} - Mg\cos\theta\,\hat{j}} \quad (40)$$

$$\boxed{\vec{F}_{tot} = Mg\sin\theta\,\hat{i} + (n - Mg\cos\theta)\hat{j}} \quad (41)$$

By Newton II $\quad \boxed{\vec{F}_{tot} = M\vec{a} = M(a_x\hat{i} + a_y\hat{j})} \quad (42)$

We know that $a_y = -\dfrac{v(\theta)^2}{R}$

So $\quad F_{tot,y} = -M\dfrac{v(\theta)^2}{R}$

$$\Rightarrow \quad \boxed{n - Mg\cos\theta = -2Mg(1-\cos\theta)} \quad (43)$$

$$\boxed{n = Mg\left[3\cos\theta - 2\right]} \quad (44)$$

Clearly at $\theta = 0$ $\cos\theta = 1$ $n = Mg$. As θ increases, $\cos\theta$ decreases, and n decreases.

When $n \to 0$ the block is barely in contact with the sphere. This is the angle beyond which it flies off the sphere

$$\Rightarrow \boxed{3\cos\theta_{max} - 2 = 0}$$ 45

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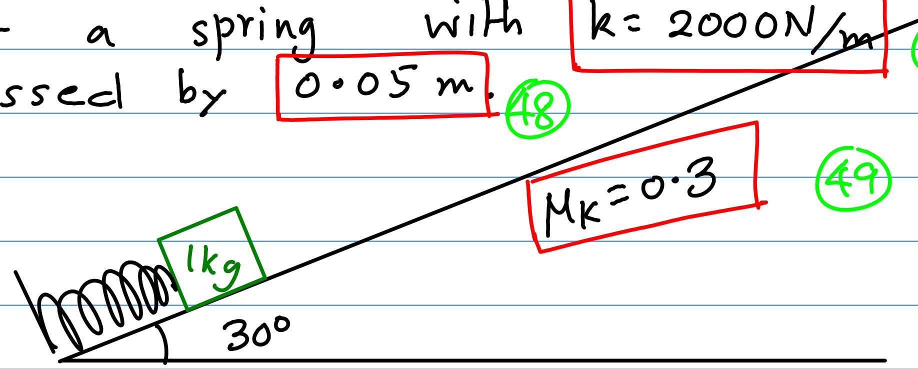
$$\cos\theta_{max} = \frac{2}{3} \Rightarrow \boxed{\begin{aligned}\theta_{max} &= 0.841 \text{ rad} \\ &= 48.2^\circ\end{aligned}}$$

Example 3: A block of mass 1kg is held against a spring with $\boxed{k = 2000 N/m}$ 47 compressed by $\boxed{0.05 \text{ m}}$. 48



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If $\mu_K = 0.3$, how far up the slope does it go from its starting point?

Let us consider its initial position to be zero height. If it moves a distance x along the slope upwards the height it gains is

$$\boxed{h = x \sin 30^\circ}$$ 50

$v_i = 0 \qquad h_i = 0 \quad \Rightarrow \quad \boxed{K_i = 0}$ 51 $\boxed{U_{g,i} = 0}$ 52

The spring is compressed $\Rightarrow$ $\boxed{U_{s,i} = \frac{1}{2} k x_i^2}$ 53

$$= \frac{1}{2} \times 5000 \frac{N}{m} \times (5\times10^{-2})^2 = 6.25\ J$$

$\Rightarrow$ $\boxed{E_{mech,i} = 6.25\ J}$ 54

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Let the final height be $\boxed{h_f = x_f \sin 30^\circ = \frac{x_f}{2}}$

Since it is at rest $\boxed{v_f = 0}$ 56 $\Rightarrow$ $\boxed{K_f = 0}$ 57

The spring is not distorted in the final state

$\boxed{U_{s,f} = 0}$ 58

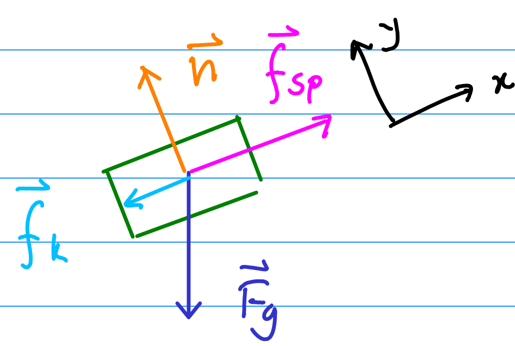
$\boxed{U_{g,f} = mgh_f = mg\frac{x_f}{2}}$ 59

$\boxed{E_{mech,f} = mg\frac{x_f}{2}}$ 60

Now, we do have a nonconservative force, f_k

So $\boxed{W_{nc} = E_{mech,f} - E_{mech,i}}$ 61

To find f_k draw a FBD



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$\boxed{\vec{F}_g = -mg\sin\theta\,\hat{i} - mg\cos\theta\,\hat{j}}$

$$a_y = 0 \Rightarrow \quad F_{tot,y} = n - mg\cos\theta = 0$$

$$\Rightarrow \boxed{n = mg\cos\theta} \quad 63$$

$$\Rightarrow \boxed{f_k = \mu_k mg\cos\theta} \quad 64$$

$$\boxed{W_{nc} = -f_k \Delta x = -f_k x_f = -\mu_k mg\cos\theta\, x_f} \quad 65$$

So applying the W-Energy theorem

$$-\mu_k mg\cos\theta\, x_f = mg\, x_f \sin\theta - 6.25\text{ J}$$

or

$$1\text{ kg} \times 9.8\text{ m/s}^2\, x_f \left[\tfrac{1}{2} + 0.3 \times \tfrac{\sqrt{3}}{2}\right] = 6.25\text{ J}$$

$$\boxed{x_f = 0.84\text{ m}} \quad 66$$