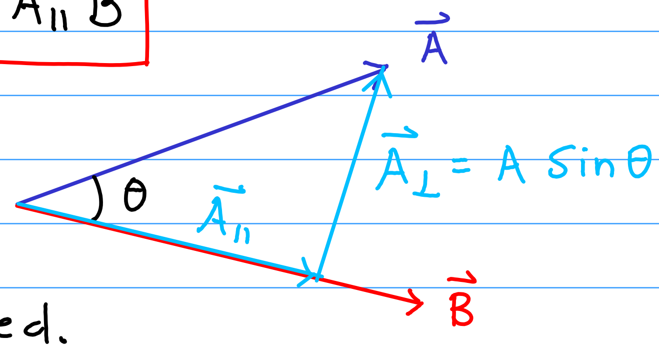
The Cross Product

We have already encountered one type of product of two vectors, the dot product or scalar product

$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta = A_{\parallel} B}$$



This produces a scalar, which does not depend on how the axes are oriented.

There is another important product between two vectors, called the cross or vector product which results in a vector.

$$\boxed{\vec{C} = \vec{A}\times\vec{B}}$$

$\bar{C}$ can be specified by giving its magnitude and direction.

$$\boxed{|\vec{C}| = |\vec{A}|\,|\bar{B}|\,\sin\theta = A_{\perp} B}$$

The direction of $\bar{C}$ is given by the Right Hand Rule.

Any two vectors $\vec{A}, \bar{B}$ that are not parallel define a plane. $\bar{C}$ is perpendicular to this plane. There are two "sides" to any plane, and thus two perpendiculars.

To choose the correct $\perp$ curl the fingers of your right hand from $\vec{A}$ to $\vec{B}$. Your extended thumb points to the direction of $\vec{C}$

Apply this set of rules to the unit vectors $\hat{i}, \hat{j}, \hat{k}$

$\hat{i} \times \hat{i} = 0$	$\hat{i} \times \hat{j} = \hat{k}$	$\hat{i} \times \hat{k} = -\hat{j}$
$\hat{j} \times \hat{i} = -\hat{k}$	$\hat{j} \times \hat{j} = 0$	$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$	$\hat{k} \times \hat{j} = -\hat{i}$	$\hat{k} \times \hat{k} = 0$

Note that $\vec{A} \times \vec{B}$ is antisymmetric

$$\boxed{\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}}$$

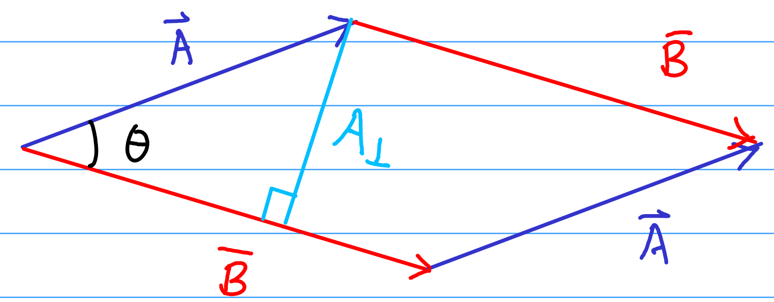
Algebraically if

$$\boxed{\begin{aligned} \vec{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \\ \vec{B} &= B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \end{aligned}}$$

and

$$\boxed{\vec{A}\times\vec{B} = \hat{i}\,(A_y B_z - A_z B_y) + \hat{j}\,(A_z B_x - A_x B_z) + \hat{k}\,(A_x B_y - A_y B_x)}$$

The magnitude of $\vec{A}\times\vec{B}$ is the area of the parallelogram made by $\vec{A}$ and $\vec{B}$



Area = Base x height = $A_\perp\, B$ = $AB\sin\theta$

A useful identity is the cyclic triple product

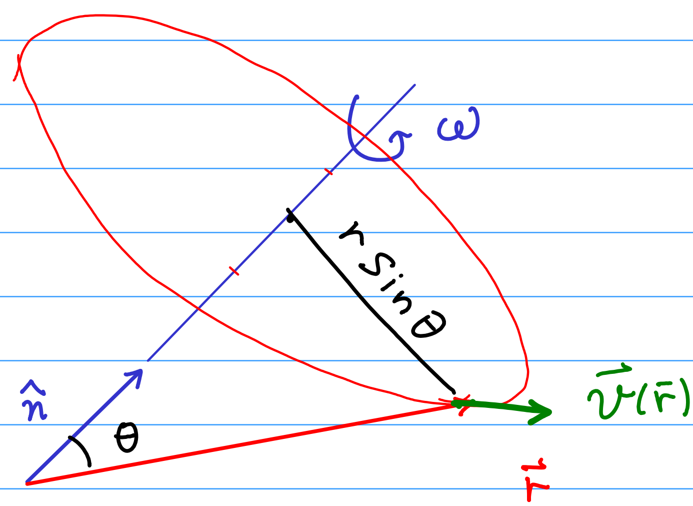
$$\boxed{\begin{aligned}&\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})\\ &= A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x)\end{aligned}}$$

The cross product has an enormous number of applications in physics.

Our first application is to find the velocity of a point in a rotating rigid body

Suppose the axis of rotation is $\hat{n}$ (a unit vector) and the fixed point of the body is the origin. Then the rotational velocity of the point $\bar{r}$ is

$$\boxed{\vec{v}_{rot}(\bar{r}) = \omega\, \hat{n} \times \vec{r}}$$



Clearly $\boxed{|\vec{v}_{rot}(\bar{r})| = \omega r \sin\theta = \omega \times \text{distance from axis.}}$

Also $\vec{v}_{rot}$ should be $\perp$ to the plane of $\hat{n}$ and $\vec{r}$ so indeed it is

$$\bar{v}_{rot}(\bar{r}) = \omega\, \hat{n} \times \vec{r}$$

Now consider a body that is rotating while its CM is moving at $\vec{v}_{CM}$.

Let us assume that the origin O is the CM of the body. Then

$$\boxed{\vec{v}_{tot}(\bar{r}) = \vec{v}_{CM} + \vec{v}_{rot}(\bar{r}) = \vec{v}_{CM} + \omega\, \hat{n} \times \bar{r}}$$

The KE is $\displaystyle\int \frac{1}{2} d^3\bar{r}\; \rho(\bar{r}) \left(\vec{v}_{tot}(\bar{r})\right)^2$

$$= \frac{1}{2}\int d^3\bar{r}\; \rho(\bar{r}) \left[\vec{v}_{CM}^{\,2} + 2\bar{v}_{CM}\cdot\bar{v}_{rot}(\bar{r}) + \vec{v}_{rot}^{\,2}(\bar{r})\right]$$

In the 1st term $\vec{v}_{CM}^{\,2}$ is independent of $\bar{r}$ and

$$\boxed{\int \rho(\bar{r})\, d^3\bar{r} = M} \quad \text{(total mass)}$$

$$\boxed{\text{1st term} = \frac{1}{2} M \vec{v}_{CM}^{\,2} = \text{Translational KE}}$$

$$\boxed{\text{Last term} = \frac{1}{2}\int \rho(\bar{r})\, d^3\bar{r}\;\; \omega^2 \delta(\bar{r})^2}$$

$\delta(\bar{r})$ = distance from axis of rotation

$$\boxed{\int d^3\bar{r}\; \rho(\bar{r}) (\delta(\bar{r}))^2 = I} \quad \text{(moment of inertia)}$$

$$\text{Last term} = \boxed{\frac{1}{2} I \omega^2 = \text{Rotational KE}}$$

What about the cross-term?

$$\boxed{\int d^3\bar{r}\; \rho(\bar{r})\; \bar{v}_{CM}\cdot\vec{v}_{rot}(\bar{r}) = \omega \int d^3\bar{r}\; \rho(\bar{r})\; \bar{v}_{CM}\cdot(\hat{n}\times\bar{r})}$$

Use the triple product identity

$$\boxed{\bar{v}_{CM} \cdot (\hat{n} \times \bar{r}) = \bar{r} \cdot (\bar{v}_{CM} \times \hat{n})}$$

$\bar{v}_{CM}$ and $\hat{n}$ are independent of $\bar{r}$, so take them out of the integral

cross-term $$\boxed{(\bar{v}_{CM} \times \hat{n}) \cdot \underbrace{\int d^3r \; \rho(\bar{r})\, \bar{r}}_{M\vec{R}_{CM}}}$$

However, we have chosen the CM as the origin. So $\boxed{\vec{R}_{CM} = 0}$

$$\Rightarrow \quad \boxed{K = \frac{1}{2} M \vec{v}_{CM}^{\,2} + \frac{1}{2} I \omega^2}$$

Note: This only works if we choose the CM as the fixed point.