Gravitational Energy, Escape speed, and orbital mechanics

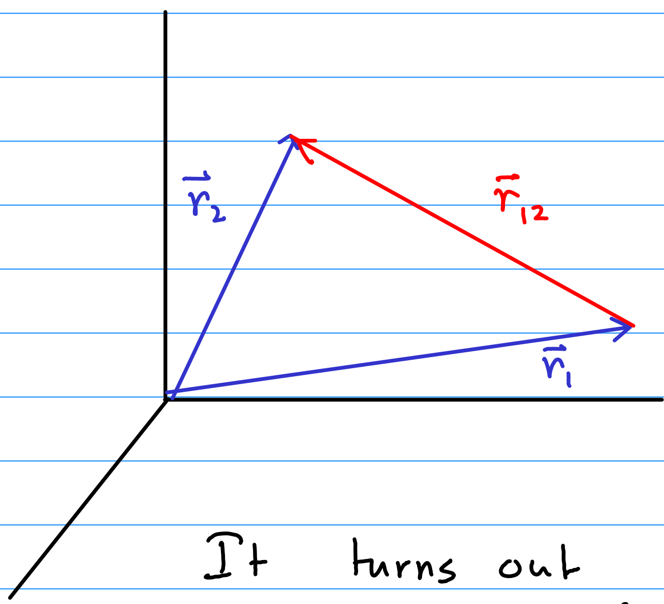
Let us once again consider two masses M_1 and M_2 at locations $\vec{r}_1$, $\vec{r}_2$.

$$\boxed{\vec{F}_{12} = - \frac{G M_1 M_2}{r_{12}^2} \hat{r}_{12}} \quad (1)$$

$$\boxed{G = 6.67 \times 10^{-11} \frac{N m^2}{kg^2}} \quad (2)$$

$\vec{F}_{12}$ = force of 1 on 2

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 \qquad \hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$$



It turns out that this force is conservative. Recall for a conservative force

$$\boxed{\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} \quad \text{is independent of path}} \quad (3)$$

and can be written in terms of a potential energy

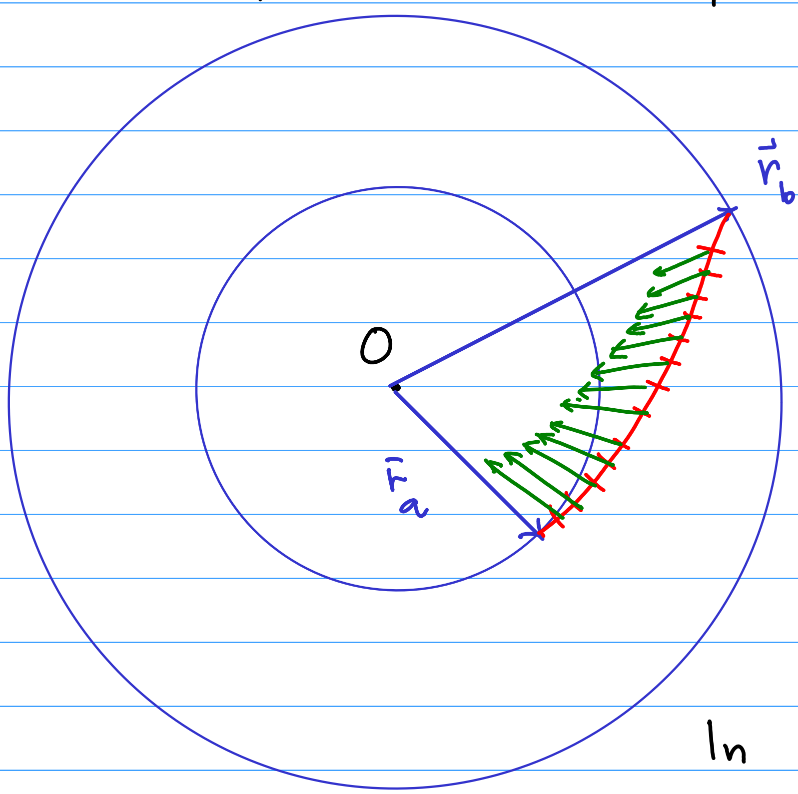
$$\boxed{\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = -\left(U(\vec{r}_b) - U(\vec{r}_a)\right)} \quad (4)$$

Let us simplify the situation by keeping M_1 fixed and choosing the origin at $\vec{r}_1$

Then the force on body 2, which I will call $\vec{F}$ is

$$\boxed{\vec{F} = -\frac{G M_1 M_2}{r^2} \hat{r}} \quad (5)$$

$\hat{r} = \hat{e}_r =$ unit vector pointing radially outwards



As usual, divide the path into tiny segments.

The green arrows are the force of M_1 on M_2 pointing radially inwards.

In $\vec{F} \cdot d\vec{r}$, clearly only the radial part of the path matters.

So $$\boxed{\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{r_a}^{r_b} F(r)\, dr} \quad (6)$$

No vectors on RHS

$$= - \int_{r_a}^{r_b} \frac{G M_1 m_2}{r^2} dr = \frac{G M_1 M_2}{r_b} - \frac{G M_1 m_2}{r_a} \quad (7)$$

depends only on the endpoints. So we know this force is conservative, and the potential energy corresponding to it is

$$\boxed{U_g(r) = - \frac{G M_1 M_2}{r}} \quad (8)$$

The zero of this potential is when $r \to \infty$, the objects are very far from each other.

This is a bit strange, because we have previously used, near the Earth's surface

$$\boxed{U_g^{old} = Mgh.} \quad (9)$$

What is the connection? Let us take the surface of the Earth to be zero height, and thus zero potential. So our new potential energy is

$$\tilde{U}_g(h) = U_g(R_E + h) - U_g(R_E) \quad (10)$$

$$= - \frac{G M_E M}{R_E + h} + \frac{G M_E M}{R_E} \quad (11)$$

For $h << R_E$

$$\boxed{\frac{1}{R_E + h} \simeq \frac{1}{R_E} \left[1 - \frac{h}{R_E} + \frac{h^2}{R_E^2} + \cdots \right]}$$

$$\boxed{\tilde{U}_g(h) = -\frac{GM_E M}{R_E}\left\{1 - \frac{h}{R_E} + \frac{h^2}{R_E^2} + \cdots\right\} + \frac{GM_E M}{R_E}} \quad (12)$$

$$\boxed{= \underbrace{\frac{GM_E M}{R_E^2}}_{=g}\, h + \text{small terms}} \quad (13)$$

$$\boxed{= Mgh + \text{small}} \quad (14)$$

So the form Mgh only works near the Earth's surface.

This leads us to the notion of Escape speed. Suppose a Mass M has a speed v_0 at the Earth's surface. Let us ignore all forces except gravity and imagine that it is going radially away from the Earth initially.

Since all forces are conservative $\Delta E_{mech} = 0$

$$\boxed{E_{mech,i} = \frac{1}{2}Mv_0^2 - \frac{GM_E M}{R_E}} \quad (15)$$

Let the farthest point away from the Earth it reaches be r_{max}. At that point it is momentarily at rest.

So $$\boxed{E_{mech,f} = -\frac{GM_E M}{r_{max}}} \quad (16)$$

$$\boxed{\Delta E_{mech} = 0 \Rightarrow -\frac{GM_E \cancel{M}}{r_{max}} = \frac{1}{2}\cancel{M}v_0^2 - \frac{GM_E\cancel{M}}{R_E}}$$ (17)

or $$\boxed{\frac{1}{r_{max}} = \frac{1}{R_E} - \frac{v_0^2}{2GM_E}}$$ (18)

Of course, for this to make sense the RHS must be positive.

As v_0 increases the RHS becomes smaller and smaller, and thus r_{max} gets bigger and bigger.

If $$\boxed{\frac{1}{R_E} = \frac{v_0^2}{2GM_E}}$$ (19) $r_{max} = \infty$

The body completely escapes the Earth's gravitational field.

$$\boxed{v_{esc}^2 = \frac{2GM_E}{R_E} = \frac{2\times 6.67\times 10^{-11}\times 6\times 10^{24}}{6.4\times 10^6} = 1.25\times 10^8\ (m/s)^2}$$ (20)

$$\Rightarrow \boxed{v_{esc} = 1.12\times 10^4\ m/s = 11.2\ km/sec}$$ (21)

If the initial speed is greater than v_{esc}, the particle has a nonzero KE at ∞.

Example: What is the Kinetic, Potential, and total mechanical energy of a Geostationary satellite of mass $\boxed{M = 500\,kg}$? (22)

Geostationary means if it sits above the Equator it remains above the same place. So its orbital period around the Earth = 1 day

$$\boxed{T = 86400\,sec} \quad (23)$$

What is the radius at which it is geostationary?
Let it be r_0

Earth's force $\boxed{F = \dfrac{G M_E M}{r_0^2} = \dfrac{M v^2}{r_0}}$ (24)

$$\Rightarrow \quad \boxed{v = \sqrt{\frac{G M_E}{r_0}}} \quad (25)$$

and $\boxed{T = \dfrac{2\pi r_0}{v} = \dfrac{2\pi}{\sqrt{G M_E}}\, r_0^{3/2}}$ (26)

$$\boxed{T^2 = \frac{4\pi^2}{G M_E} r_0^3} \quad (27)$$

(28)

So $$\boxed{\begin{aligned} r_0 &= \left\{ \frac{G M_E T^2}{4\pi^2} \right\}^{\frac{1}{3}} = \left\{ \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 8.64^2 \times 10^8}{4\pi^2} \right\}^{\frac{1}{3}} \\ &= \left\{ 75.67 \times 10^{21} \right\}^{1/3} = 4.23 \times 10^7\,m \end{aligned}}$$

29

and the speed is $\boxed{v = \sqrt{\dfrac{GM_E}{r_0}} = 3.08 \times 10^3 \, m/s}$

A convenient number is $\boxed{GM_E = 4 \times 10^{14} \, Nm^2/kg}$ 30

The gravitational PE is

$$\boxed{U_g(r_0) = -\frac{GM_E M}{r_0} = -\frac{4 \times 10^{14} \times 500}{4.23 \times 10^7} = -4.73 \times 10^9 \text{ Joules}}$$ 31

Again, the $-$ sign reflects the fact that the zero of PE is when $r \to \infty$

What is the KE?

$$K = \frac{1}{2} M v^2$$

But from 24 $\boxed{Mv^2 = \dfrac{GM_E}{r_0}}$ 32

33

$$\Rightarrow \boxed{K = +\frac{GM_E}{2r_0} = -\frac{U_g(r_0)}{2} = +2.36 \times 10^9 \text{ Joules}}$$

$$\boxed{E_{mech}(orbit) = K + U_g = -2.36 \times 10^9 \text{ Joules}}$$ 34

What was the satellite's mechanical energy when it was sitting at rest in the lab where it was made? Then it only had PE, and it was at $r = R_E$ so

$$\boxed{\begin{aligned} E_{mech}\text{(before launch)} &= -\frac{G M_E M}{R_E} \\ &= -\frac{4\times10^{14} \times 500}{6.4\times10^{6}} = -3.125\times10^{10}\ \text{Joules} \end{aligned}}$$ 35

So the nonconservative work done to launch the satellite and place it in a geostationary orbit is

$$W_{nc} = E_{mech}\text{(orbit)} - E_{mech}\text{(before launch)}$$

$$= -2.36\times10^{9}\,\text{J} - (-3.125\times10^{10}\,\text{J})$$

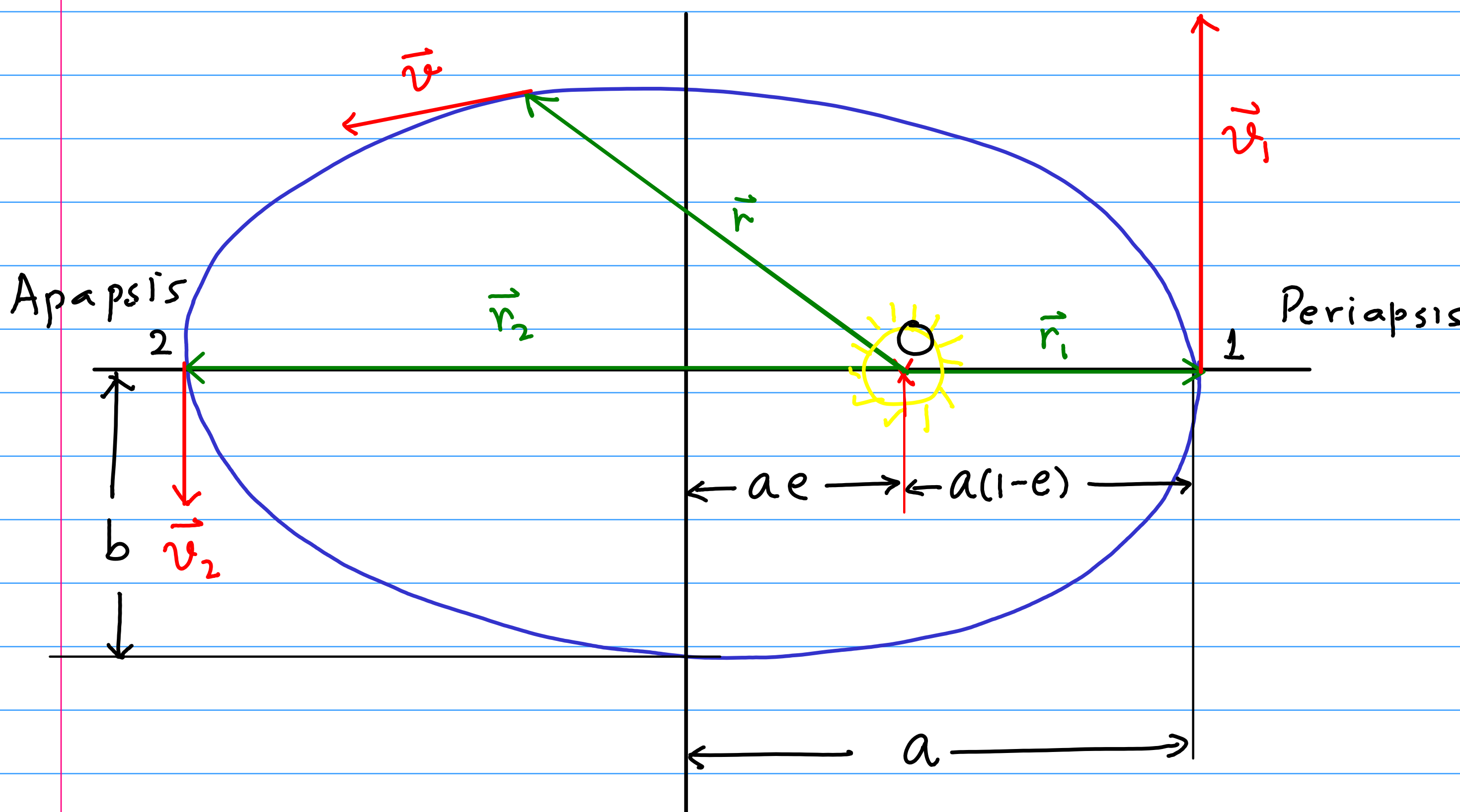
$$\boxed{W_{nc} = 2.89\times10^{10}\ \text{Joules}}$$ 36

A humongous amount of energy! All this energy has to come from the rocket that launches it from the Earth's surface.

So far we have focused on the mechanical energy, which is conserved.

Now, let's take a look at the angular momentum, which is also conserved, because the force is central.

Let us go back to the elliptical orbit. Choose the origin at the focus where the Sun is.



In general $\boxed{\vec{L} = M\,\vec{r}\times\vec{v}}$ (37)

and usually $\vec{v}$ is not $\perp$ to $\vec{r}$. However, there are two points where $\vec{v} \perp \vec{r}$. Those are

$$\boxed{\begin{array}{l} \text{periapsis} = \text{point of closest approach} = \vec{r}_1 \\ \text{apapsis} = \text{point furthest from } O = \vec{r}_2 \end{array}}$$ (38)

If the central object is the Sun, these points are called perihelion and aphelion. If we consider satellites orbiting the Earth, these points are called perigee and apogee.

If $\vec{v}_1$ is the velocity at 1 and $\vec{v}_2$ at 2, then, by the conservation of $\vec{L}$

$$\boxed{M r_1 v_1 = M r_2 v_2} \quad (39)$$

$$\Rightarrow \quad \boxed{\frac{r_2}{r_1} = \frac{v_1}{v_2}} \quad (40)$$

Suppose we know r_1, v_1 by observing the satellite when it is at perigee. We also know that mechanical energy must be conserved. So, it must be true that

$$\boxed{\frac{1}{2} M v_1^2 - \frac{G M_E M}{r_1} = \frac{1}{2} M v_2^2 - \frac{G M_E M}{r_2}} \quad (41)$$

Cancel M and multiply by 2

$$\boxed{v_1^2 - \frac{2 G M_E}{r_1} = v_2^2 - \frac{2 G M_E}{r_2}} \quad (42)$$

Since v_1, r_1 are known the LHS is known

We also know $\boxed{v_2 = \frac{r_1 v_1}{r_2}}$ (43) so we can solve for both r_2 and v_2. Let us see this explicitly in an example.

Example: Moving a satellite from one orbit to another.

Consider a satellite in circular Earth orbit with a period of $\boxed{6 \text{ hours} = 21600 \text{ sec.}}$ 44

First let us find the r_0 and v_0 of the circular orbit

$$\boxed{\frac{M v_0^2}{r_0} = \frac{G M_E M}{r_0}} \; 45 \quad \Rightarrow \quad \boxed{v_0 = \sqrt{\frac{G M_E}{r_0}}} \; 46$$

$$\boxed{T_0 = \frac{2\pi r_0}{v_0} = \frac{2\pi}{\sqrt{G M_E}} \, r_0^{3/2}} \; 47$$

$$r_0 = \left\{ \frac{G M_E T_0^2}{4\pi^2} \right\}^{1/3} = \left\{ \frac{4 \times 10^{14} \times 2.16^2 \times 10^8}{4\pi^2} \right\}^{\frac{1}{3}}$$

$$\boxed{r_0 = \left\{ 4.73 \times 10^{21} \right\}^{1/3} = 1.68 \times 10^7 \, m} \; 48$$

and $\boxed{v_0 = \frac{2\pi r_0}{T_0} = 4.88 \times 10^3 \, m/s}$ 49

Now the satellite is given a boost of $\boxed{1000 \, m/s}$ 50 along its direction of motion.

Since $\boxed{v = 5.88 \times 10^3 \, m/s}$ 51 is still $\perp$ to $\bar{r}$ this must be the perigee of the new orbit

$$\boxed{r_1 = r_0 = 1.68 \times 10^7 \, m} \; 52 \qquad \boxed{v_1 = 5.88 \times 10^3 \, m/s} \; 53$$

Call the distance and speed at apogee r_2 and v_2

By conservation of angular momentum

$$\boxed{r_2 v_2 = r_1 v_1 = 1.68 \times 10^7 \times 5.88 \times 10^3 = 9.88 \times 10^{10} \ m^2/s}$$

54

By conservation of Emech

55

$$\boxed{\frac{E_{mech,1}}{M} = \frac{1}{2} v_1^2 - \frac{GM_E}{r_1} = \frac{1}{2} v_2^2 - \frac{GM_E}{r_2} = \frac{E_{mech,2}}{M}}$$

$$LHS = \frac{1}{2} \times (5.88 \times 10^3)^2 - \frac{4 \times 10^{14}}{1.68 \times 10^7} =$$

$$\boxed{\frac{E_{mech,1}}{M} \quad 1.73 \times 10^7 - 2.38 \times 10^7 = -6.5 \times 10^6 \ \frac{J}{kg}}$$

Now replace, on the RHS, by using 54 56

$$\boxed{\frac{1}{r_2} = \frac{v_2}{r_1 v_1} = \frac{v_2}{9.88 \times 10^{10} \ m^2/s}}$$ 57

So $$\boxed{\frac{1}{2} v_2^2 - \frac{4 \times 10^{14} \ v_2}{9.88 \times 10^{10}} = -6.5 \times 10^6}$$ 58

Let $\boxed{v_2 = 10^3 \ \tilde{v}_2}$ 59

$$\Rightarrow \boxed{0.5 \times 10^6 \ \tilde{v}_2^2 - 4.05 \times 10^6 \ \tilde{v}_2 = -6.5 \times 10^6}$$ 60

$$\Rightarrow \quad \tilde{v}_2^2 - 8.1\,\tilde{v}_2 + 13 = 0$$

$$\tilde{v}_2 = \frac{8.1 \pm \sqrt{(8.1)^2 - 52}}{2} = \frac{8.1 \pm 3.69}{2} \begin{array}{l} \nearrow 5.89 \\ \searrow 2.2 \end{array}$$

So the two possible answers for v_2 are

$$\boxed{v_2 = 5.89 \times 10^3 \text{ m/s}} \quad 61$$

$$\text{or } \boxed{v_2 = 2.2 \times 10^3 \text{ m/s}} \quad 62$$

The 1st answer is just v_1 and so we must choose $v_2 = 2.2 \times 10^3$ m/s

This allows us to find

$$\boxed{r_2 = \frac{r_1 v_1}{v_2} = \frac{9.88 \times 10^{10}}{2.2 \times 10^3} = 4.49 \times 10^7 \text{ m}} \quad 63$$

Now, from the figure, it is clear that

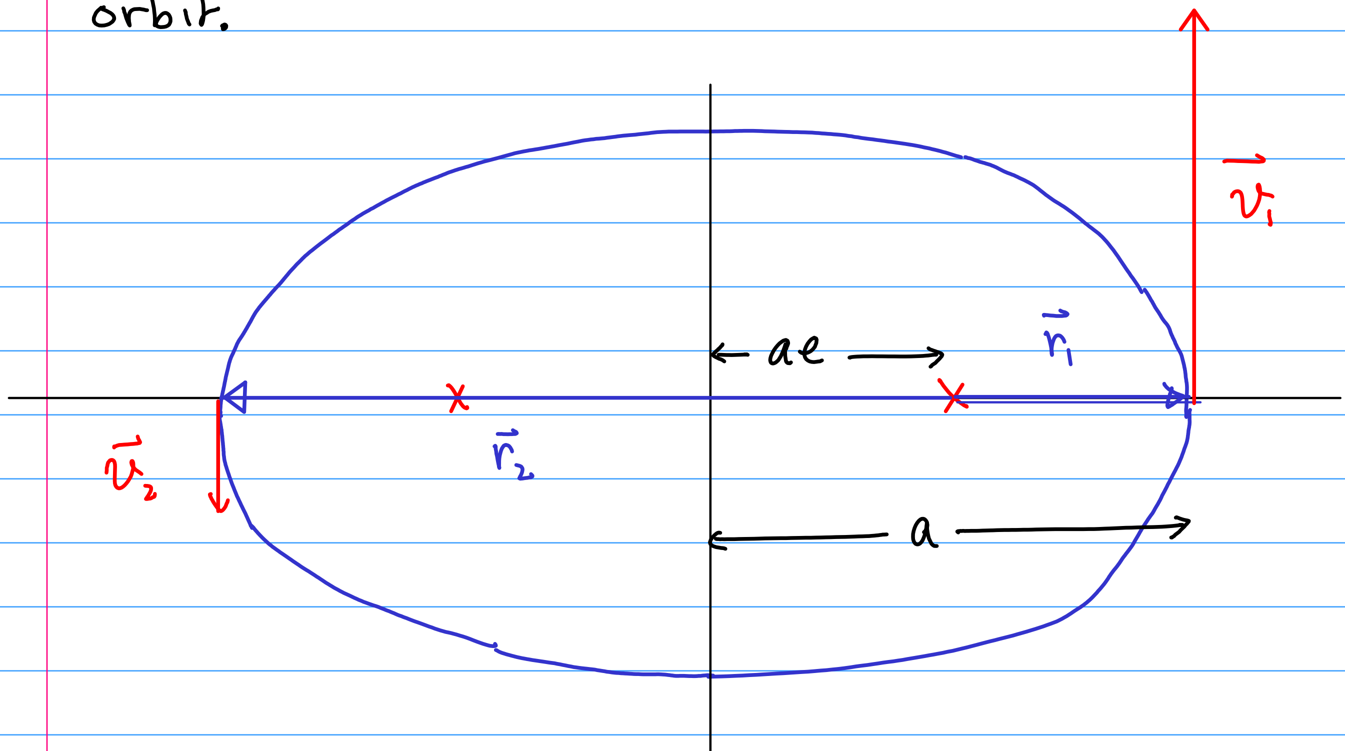
$$\boxed{r_1 = a(1-e)} \quad 64 \qquad \text{and} \qquad \boxed{r_1 + r_2 = 2a} \quad 65$$

$$\text{So} \quad \boxed{a = \frac{r_1 + r_2}{2} = 3.09 \times 10^7 \text{ m}} \quad 66$$

$$\text{and} \quad \boxed{\frac{r_1}{a} = 1 - e = 0.544} \quad 67$$

$\Rightarrow$ $\boxed{e = 0.456}$ is the eccentricity of the orbit. 68

It turns out that if one knows the perigee and apogee distances r_1, r_2 one can figure out everything else about the orbit.



$$\boxed{E_{mech} = \frac{1}{2} M v_1^2 - \frac{G M_E M}{r_1} = \frac{1}{2} M v_2^2 - \frac{G M_E M}{r_2}} \quad (69)$$

define

$$\boxed{\tilde{E} = \frac{2 E_{mech}}{M}} \quad (70)$$

This has to be negative for a bound orbit else the satellite has greater than escape speed and will escape.

$$\boxed{\tilde{E} = -|\tilde{E}| = v_1^2 - \frac{2 G M_E}{r_1} = v_2^2 - \frac{2 G M_E}{r_2}} \quad (71)$$

Angular momentum is conserved

$$\boxed{L = M r_1 v_1 = M r_2 v_2} \quad (72)$$

define $\boxed{\tilde{L} = \frac{L}{M} = r_1 v_1 = r_2 v_2}$ (73)

So $\boxed{v_1 = \frac{\tilde{L}}{r_1}}$ (74)

$\Rightarrow \boxed{\tilde{E} = \frac{\tilde{L}^2}{r_1^2} - \frac{2GM_E}{r_1}}$ (75) $\Rightarrow \boxed{-|\tilde{E}|\, r_1^2 = \tilde{L}^2 - 2GM_E r_1}$ (76)

or $|\tilde{E}|\, r_1^2 - 2GM_E r_1 + \tilde{L}^2 = 0$

$$\boxed{r_\pm = \frac{GM_E \pm \sqrt{(GM_E)^2 - \tilde{L}^2 |\tilde{E}|}}{|\tilde{E}|}} \quad (77)$$

In fact $\boxed{r_+ \text{ is } r_2}$ (78) (apogee) and $\boxed{r_- \text{ is } r_1}$ (79) (perigee)

So $\boxed{r_1 + r_2 = \frac{2GM_E}{|\tilde{E}|} = 2a}$ (80) $\Rightarrow \boxed{|\tilde{E}| = \frac{GM_E}{a}}$ (81)

or $\boxed{\tilde{E} = -\frac{GM_E}{a}}$ (82)

So if we know r_1, r_2 we go to

$$\tilde{E} = v_1^2 - \frac{2GM_E}{r_1} = -\frac{GM_E}{a}$$

$\Rightarrow \boxed{v_1^2 = GM_E\left(\frac{2}{r_1} - \frac{1}{a}\right)}$ (83) $\boxed{v_2^2 = GM_E\left(\frac{2}{r_2} - \frac{1}{a}\right)}$ (84)

So we can find the speeds v_1, v_2 as well