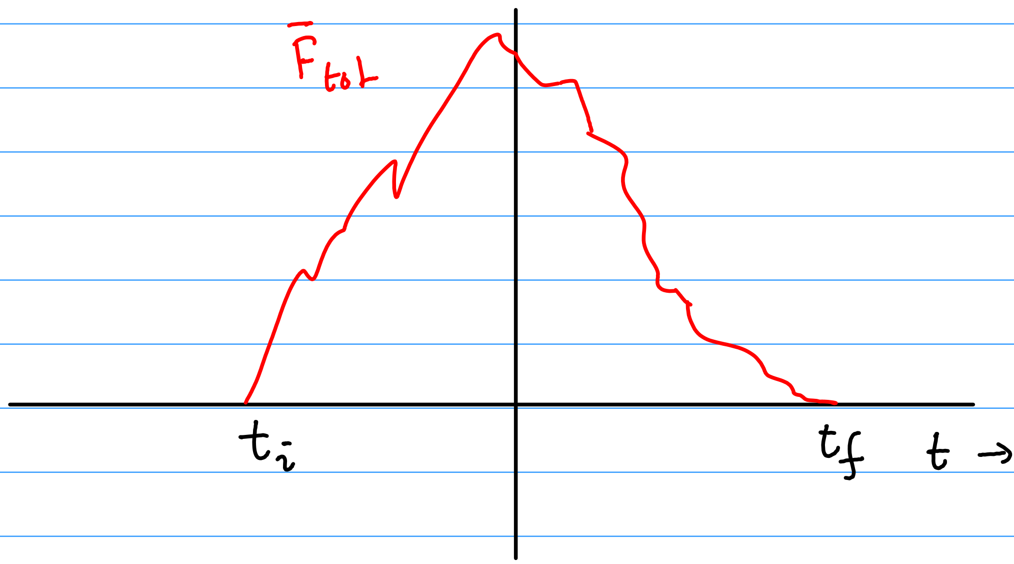
The Impulse-Momentum Theorem and 1D Collisions

Often, as in collisions, the forces that act on a body are large but of short duration.

An example is shown below



While the detailed effect of this force is complicated it has a simple effect on the object's momentum

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{tot}(t)} \quad (1)$$

$$\boxed{\Delta \vec{p} = \int_{t_i}^{t_f} \vec{F}_{tot}(t)\, dt = \vec{I} \equiv \text{Impulse}} \quad (2)$$

This is the Impulse-Momentum Theorem

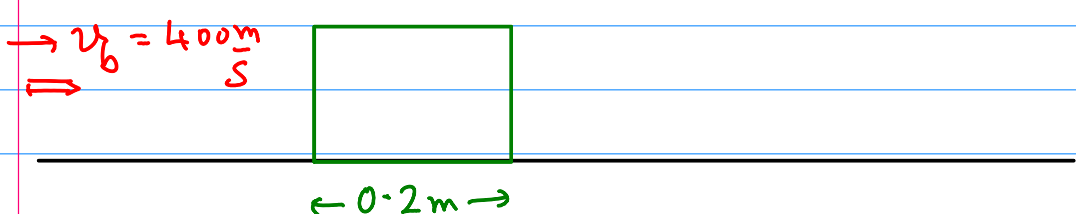
Compare to the Work-KE Theorem

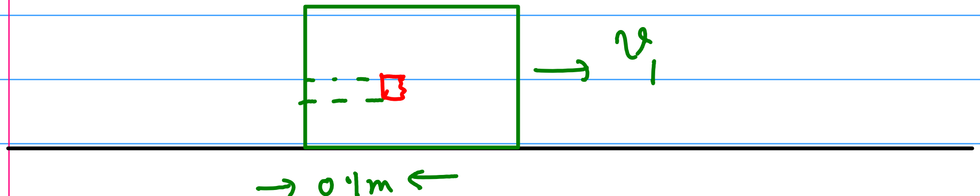
$$\boxed{W_{tot} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{tot} \cdot d\vec{r} = \Delta K = K_f - K_i} \tag{3}$$

$$\boxed{\vec{I} = \int_{t_i}^{t_f} \vec{F}_{tot} \, dt = \Delta \vec{p} = \vec{p}_f - \vec{p}_i} \tag{4}$$

In collisions, usually even though there is friction, during the time the collision is taking place the frictional impulse can be neglected compared to the collisional impulse.

Example 1: A block of mass 5kg sits on a horizontal surface with $\mu_k = 0.5$. A bullet of mass 0.05 kg and velocity 400 m/s slams into the block and embeds itself halfway into the block.





Initial momentum of bullet + block

$$\boxed{\begin{aligned} p_{i,tot} &= 0.05\,kg \times 400\,\frac{m}{s} + 5\,kg \times 0\,\frac{m}{s} \\ &= 20\,kg\,m/s \end{aligned}}$$ 5

Ignoring friction, no external forces act, so

$$\boxed{p_{f,tot} = 5.05\,kg\,v_1 = p_{i,tot} = 20\,kg\,m/s}$$ 6

$$\Rightarrow \boxed{v_1 = 3.96\,m/s}$$ 7

Now let us estimate the Impulse due to friction. The time of collision Δt can be estimated as follows: Average acceleration of bullet is 8

$$\boxed{a_{av,b} = \frac{v_1 - v_b}{\Delta t} \Rightarrow \Delta t = \frac{v_1 - v_b}{a_{av,b}}}$$

During this time the average velocity is

$$\boxed{\frac{v_1 + v_b}{2}}$$ 9

$$\Rightarrow \boxed{\Delta x_b = \frac{v_1 + v_b}{2}\,\frac{v_1 - v_b}{a_{av,b}} \Rightarrow \frac{v_1^2 - v_b^2}{2a_{av,b}}}$$ 10

11

$$\Rightarrow \boxed{a_{av,b} = \frac{v_1^2 - v_b^2}{2\,\Delta x_b} = \frac{(3.96)^2 - 400^2}{2 \times 0.1} = -8 \times 10^5\,m/s^2}$$

$$\Rightarrow \boxed{\Delta t = \frac{v_1 - v_b}{a_{av,b}} = 4.95 \times 10^{-4}\,sec}$$ 12

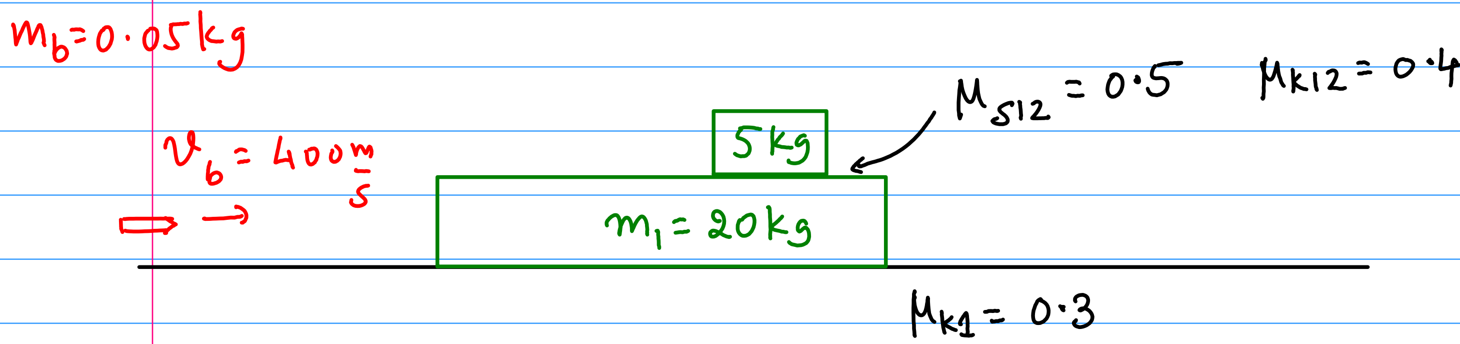
$$\boxed{F_f = \mu_K m g = 0.5 \times 5\,kg \times 10\, m/s^2 = 25\,N} \quad (13)$$

Impulse of F_f during the collision is

$$\boxed{I_f = F_f \,\Delta t = 1.25 \times 10^{-3}\ kg\,m/s} \quad (14)$$

Negligible compared to the $20\,kg\,m/s$ Impulse due to the collisional impulse of the bullet.

Example 2



The collision takes 0.005 sec. Does the top block slide with respect to the bottom block or do they move together? If it slides how far does it slide back relative to the bottom block until they start sliding together?

Assume that they move together. Then

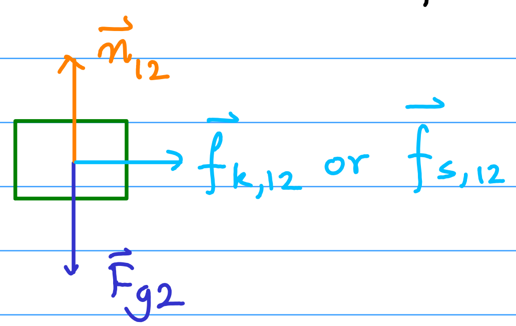
$$\boxed{\begin{aligned} p_{i,tot} &= 0.05\,kg \times 400\,m/s = 20\,kg\,m/s = p_{f,tot} \\ &= 25.05\,kg\ v_f \end{aligned}} \quad (15)$$

$$\Rightarrow \boxed{v_f = 0.7984 \text{ m/s}} \quad (16)$$

Since the collision takes 0.005 sec the average acceleration of the blocks is

$$\boxed{a_{av} = \frac{v_f}{\Delta t} = 159.7 \text{ m/s}^2} \quad (17)$$

Now consider the FBD of the top block



(18)

$$a_y = 0 \Rightarrow \boxed{n_{12} = m_2 g}$$

Since $\vec{f}_{s,12}$ or $\vec{f}_{k,12}$ is the only force accelerating this block forward. So the max force $\vec{f}_{s,12}$ can be is

(19)

$$\boxed{f_{s,12} = \mu_s n_{12} = 0.5 \times 5\text{kg} \times 10 \text{ m/s}^2 = 25 \text{ N}}$$

This can produce a maximum acceleration of

$$\boxed{\text{Max } a_{x2} = \frac{f_{s,12}}{m_2} = 5 \text{ m/s}^2} \quad (20)$$

Certainly cannot produce an acceleration of 159.7 m/s^2.

So the blocks do not move together.

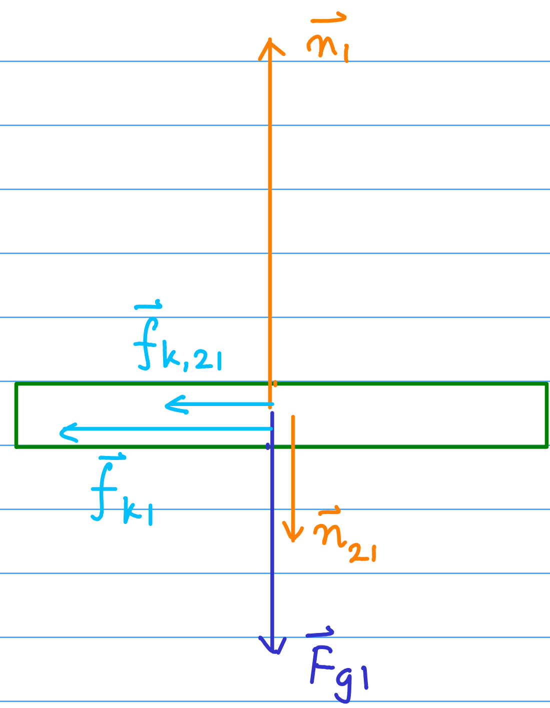
OK, let's regroup. Block 1 absorbs the impulse from the bullet and starts moving just after the collision, while block 2 remains stationary.

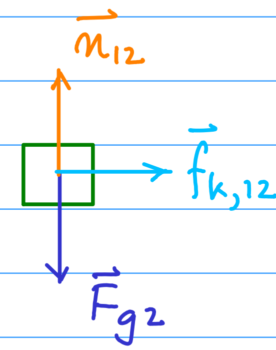
The initial velocity of block 1 is

$$\boxed{I = 20\,kg\,m/s = 20.05\,kg \cdot v_{1i}} \quad (21)$$

$$\Rightarrow \boxed{v_{1i} = 0.9975\,m/s} \quad (22)$$

Now let us draw the FBD's





$a_{y2} = 0$ (23)

$\Rightarrow \boxed{n_{12} = m_2 g}$

$$\boxed{f_{k12} = \mu_{k12}\, n_{12} = 0.4 \times 5\,kg \times 10\,\frac{m}{s^2}} \quad (24)$$

$$= 20\,N$$

$$a_{y1} = 0 \Rightarrow \boxed{n_1 = n_{21} + m_1 g = (m_1 + m_2) g} \quad (25)$$

$$\Rightarrow \boxed{f_{k1} = \mu_{k1}(m_1 + m_2)\, g = 0.3 \times 25.05\,kg \times 10\,\frac{m}{s^2} = 75.15\,N} \quad (26)$$

$$\Rightarrow \quad \boxed{F_{tot,x,2} = m_2 a_{2x} = 20\,N} \quad 27$$

$$\boxed{a_{2x} = 4\,m/s^2} \quad 28$$

$$29 \quad \boxed{F_{tot,x,1} = m_1 a_{1x} = -20\,N - 75.15\,N = -95.15\,N}$$

$$\Rightarrow \quad \boxed{a_{1x} = -4.74\,m/s^2} \quad 30$$

So
$$\boxed{\begin{aligned} v_1(t) &= v_1(0) + a_{1x} t = 0.9975\,m - 4.74\,\frac{m}{s^2}\,t \\ v_2(t) &= v_2(0) + a_{2x} t = 4\,\frac{m}{s^2}\,t \end{aligned}} \quad 31$$

They move together when $v_1 = v_2$

$$0.9975 - 4.74\,t = 4t$$

$$\Rightarrow \quad \boxed{t = 0.1141\ sec} \quad 32$$

How far does block 2 slide relative to block 1 during this time?

$$\boxed{\begin{aligned} \Delta X_1 &= v_1(0)\,t + \frac{1}{2} a_{1x} t^2 \\ \Delta X_2 &= v_2(0)\,t + \frac{1}{2} a_{2x} t^2 \end{aligned}} \quad 33$$

$\Rightarrow$ relative displacement : $\Delta X_2 - \Delta X_1$

$$= \frac{1}{2}(a_{2x} - a_{1x})t^2 - v_1(0)t = \frac{1}{2} \times 8.74\,\frac{m}{s^2}(0.1141)^2 - 0.9975 \times 0.1141$$

$$= 0.0569 \text{ m} = \boxed{5.69 \text{ cm}}$$ 34

Now let us talk a bit more about collisions in 1D. We already know about the kind of collision where the two bodies stick together after the collision. This is an extreme kind of collision called a totally inelastic collision.

In any inelastic collision KE is lost to heat. In a totally inelastic collision the maximum possible KE is lost.

The other extreme kind of collision is a perfectly elastic collision, where both momentum and KE are conserved.

Example 1: Totally inelastic collision of equal masses

$\longrightarrow v_i$ at rest.

$\boxed{m}$ (1) $\boxed{m}$ (2)

35 $$\boxed{p_{i,tot} = m v_i}$$

36 $$\boxed{p_{f,tot} = 2 m v_f = p_{i,tot}}$$

$$\Rightarrow \boxed{v_f = \frac{v_i}{2}}$$ 37

What is the KE lost?

$$\boxed{K_i = \frac{1}{2} m v_i^2}$$ 38

39 $$\boxed{K_f = \frac{1}{2}(2m)\left(\frac{v_i}{2}\right)^2 = \frac{1}{4} m v_i^2}$$

So half the initial KE is lost.

Example 2 Perfectly elastic collision with the same initial conditions. Let the final velocities be v_{1f}, v_{2f}

$$\boxed{P_{i,tot} = mv_i} \quad (40) \qquad \boxed{P_f = m v_{1f} + m v_{2f}} \quad (41)$$

$$\boxed{mv_i = mv_{1f} + mv_{2f}} \quad (42) \Rightarrow \boxed{v_i = v_{1f} + v_{2f}} \quad (43)$$

$$\boxed{K_{i,tot} = \frac{1}{2} mv_i^2} \quad (44) \qquad \boxed{K_{f,tot} = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2} \quad (45)$$

$\Rightarrow$ If $K_{i,tot} = K_{f,tot}$

$$\boxed{\frac{1}{2} mv_i^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} mv_{2f}^2} \quad (46)$$

$$\boxed{v_i^2 = v_{1f}^2 + v_{2f}^2} \quad (47)$$

Since $v_i = v_{1f} + v_{2f}$ (42) we have from (47)

$$(v_{1f} + v_{2f})^2 = v_{1f}^2 + v_{2f}^2$$

or $$\boxed{v_{1f}^2 + v_{2f}^2 + 2v_{1f} v_{2f} = v_{1f}^2 + v_{2f}^2} \quad (48)$$

$$\Rightarrow \boxed{v_{1f} v_{2f} = 0} \quad (49) \quad : \text{So} \quad \boxed{v_{1f} = 0 \quad \text{or} \quad v_{2f} = 0} \quad (50)$$

From the physical situation we know

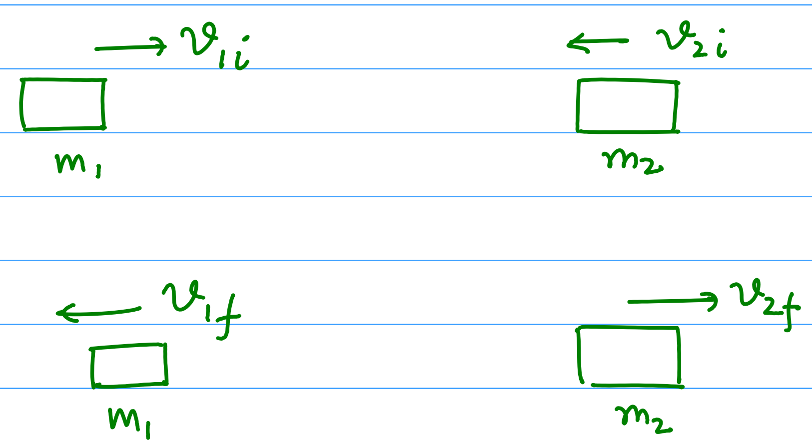
that $v_{2f} \neq 0$. So $\boxed{v_{1f} = 0}$ (S1) $\Rightarrow$ $\boxed{v_{2f} = v_{1i}}$ (S2)

For blocks of equal mass, if initially one is moving and the other is at rest, after a perfectly elastic collision the 1st mass will be at rest and the 2nd will be moving.

Let us consider a more generic case of a perfectly elastic collision

Initial

Final



Note that v_{i1}, v_{i2}, etc can be negative

We know that both momentum and kinetic energy are conserved. We know v_{1i}, v_{2i} and we want to find v_{1f} v_{2f}

(S3)

Momentum conservation $\boxed{m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}}$

KE conservation $\boxed{\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2}$

(S4)

Define $\boxed{p_{1f} = m_1 v_{1f}}$ (55) $\boxed{p_{2f} = m_2 v_{2f}}$ (56)

$\boxed{\frac{1}{2} m_1 v_{1f}^2 = \frac{p_{1f}^2}{2m_1}}$ (57) $\boxed{\frac{1}{2} m_2 v_{2f}^2 = \frac{p_{2f}^2}{2m_2}}$ (58)

So the eq^{n}s are

$$\boxed{\begin{aligned} &p_{1f} + p_{2f} = p \\ &\frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2} = K \end{aligned}} \quad (59)$$

or, multiplying by $2m_1 m_2$

$$\boxed{m_2 p_{1f}^2 + m_1 p_{2f}^2 = 2m_1 m_2 K} \quad (60)$$

$$\boxed{p_{2f} = p - p_{1f}} \quad (61)$$

$$\Rightarrow \quad m_2 p_{1f}^2 + m_1 (p - p_{1f})^2 = 2m_1 m_2 K$$

$$\text{or} \quad m_2 p_{1f}^2 + m_1 (p^2 - 2pp_{1f} + p_{1f}^2) = 2m_1 m_2 K$$

$$(m_1 + m_2) p_{1f}^2 - 2m_1 p p_{1f} + m_1 p^2 = 2m_1 m_2 K$$

$$p_{1f}^2 - 2\frac{m_1 p}{m_1+m_2} p_{1f} + \frac{m_1 p^2}{m_1+m_2} - \frac{2m_1 m_2}{m_1+m_2} K = 0$$

$$\boxed{p_{1f} = \frac{m_1 p}{m_1+m_2} \pm \sqrt{\left(\frac{m_1 p}{m_1+m_2}\right)^2 - \left(\frac{m_1 p^2}{m_1+m_2} - \frac{2m_1 m_2}{m_1+m_2} K\right)}} \quad (62)$$

Look at the argument of the square root.

$$\frac{m_1^2 p^2}{(m_1+m_2)^2} - \frac{m_1 p^2}{m_1+m_2} + \frac{2 m_1 m_2 K}{m_1+m_2}$$

$$= \frac{1}{(m_1+m_2)^2} \left\{ m_1^2 p^2 - m_1(m_1+m_2) p^2 + 2 m_1 m_2 (m_1+m_2) K \right\}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2} \left\{ -p^2 + 2(m_1+m_2) K \right\}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2} \left\{ -(p_{1i}+p_{2i})^2 + 2(m_1+m_2)\left(\frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} \right) \right\}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2} \left\{ -p_{1i}^2 - p_{2i}^2 - 2 p_{1i} p_{2i} + p_{1i}^2 + \frac{m_2}{m_1} p_{1i}^2 + p_{2i}^2 + \frac{m_1}{m_2} p_{2i}^2 \right\}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2} \left\{ \frac{m_2}{m_1} p_{1i}^2 - 2 p_{1i} p_{2i} + \frac{m_1}{m_2} p_{2i}^2 \right\}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^2} \left[\sqrt{\frac{m_2}{m_1}}\, p_{1i} - \sqrt{\frac{m_1}{m_2}}\, p_{2i} \right]^2 = \boxed{\frac{\left[m_2 p_{1i} - m_1 p_{2i} \right]^2}{(m_1+m_2)^2}} \tag{63}$$

$$\Rightarrow \boxed{p_{1f} = \frac{m_1}{m_1+m_2}(p_{1i}+p_{2i}) \pm \frac{\left[m_2 p_{1i} - m_1 p_{2i} \right]}{m_1+m_2}} \tag{64}$$

For the + sign we get

$$\boxed{p_{1f}^{(+)} = \frac{m_1}{m_1+m_2}(p_{1i}+p_{2i}) + \frac{m_2 p_{1i}}{m_1+m_2} - \frac{m_1 p_{2i}}{m_1+m_2} = p_{1i}} \quad (65)$$

So this is the case when they "pass through" each other without interacting. The correct answer is

$$p_{1f}^{(-)} = \frac{m_1}{m_1+m_2}(p_{1i}+p_{2i}) - \frac{m_2 p_{1i}}{m_1+m_2} + \frac{m_1 p_{2i}}{m_1+m_2}$$

$$\boxed{p_{1f} = \frac{m_1-m_2}{m_1+m_2}p_{1i} + \frac{2m_1}{m_1+m_2}p_{2i}} \quad (66)$$

Now go back to find p_{2f}

$$p_{2f} = p_{1i}+p_{2i}-p_{1f}$$

$$= p_{1i}+p_{2i} - \frac{m_1-m_2}{m_1+m_2}p_{1i} - \frac{2m_1}{m_1+m_2}p_{2i}$$

$$\boxed{p_{2f} = \frac{2m_2}{m_1+m_2}p_{1i} + \frac{m_2-m_1}{m_1+m_2}p_{2i}} \quad (67)$$

Let us take various limits. Suppose $\boxed{m_1=m_2}$ (68)

Then $\boxed{p_{1f} = p_{2i}}$ (69) and $\boxed{p_{2f} = p_{1i}}$ (70)

The particles exchange momenta!

Suppose $\boxed{m_2 >> m_1}$ (71) and $\boxed{p_{2i} = 0}$ (72) (heavy object is stationary)

$$p_{1f} = \frac{m_1 - m_2}{m_1 + m_2} p_{1i} + \frac{2m_1}{m_1 + m_2} p_{2i}$$

Since $m_2 \to \infty$ we ignore $\frac{m_1}{m_2}$

$$\boxed{p_{1f} \to -p_{1i}} \quad (73)$$

The light object bounces back and reverses its momentum.

This entire derivation was pretty tedious. Let us do it much more elegantly using Relativity.

The principle of Relativity says that physics should be the same in any inertial frame.

Let us go to a frame where the total momentum is zero. This is called the COM frame (center of momentum).

To go to this frame imagine that the velocities given initially v_{1i}, v_{2i} are in the stationary lab frame. You are moving with respect to the lab at v_0.

So in your frame particle 1's velocity is

$$\boxed{\begin{aligned} v'_{1i} &= v_{1i} - v_0 \\ v'_{2i} &= v_{2i} - v_0 \end{aligned}} \quad (74)$$

and

Total initial momentum in your frame is

$$\boxed{p' = p'_{1i} + p'_{2i} = m_1 v'_{1i} + m_2 v'_{2i} = p - (m_1+m_2)\, v_0} \quad (75)$$

We want this to vanish $\Rightarrow$ $\boxed{v_0 = \dfrac{p}{m_1+m_2}}$ (76)

So $\boxed{p'_i = 0}$ (77) $\Rightarrow$ $\boxed{p'_{1i} = -p'_{2i}}$ (78)

The final total momentum must be the same

$\boxed{p'_f = p'_i = 0}$ (79) $\Rightarrow$ $\boxed{p'_{1f} = -p'_{2f}}$ (80)

Now for the KE conservation

$$\boxed{K'_i = \frac{(p'_{1i})^2}{2m_1} + \frac{(p'_{2i})^2}{2m_2}} \quad (81)$$

and

$$\boxed{K'_f = \frac{(p'_{1f})^2}{2m_1} + \frac{(p'_{2f})^2}{2m_2}} \quad (82)$$

But (83) $\boxed{p'_{1i} = -p'_{2i}}$ and $\boxed{p'_{1f} = -p'_{2f}}$ (84)

$\Rightarrow$ $\boxed{K'_i = (p'_{1i})^2\left[\dfrac{1}{2m_1} + \dfrac{1}{2m_2}\right]}$ (85) $\boxed{K'_f = (p'_{1f})^2\left[\dfrac{1}{2m_1} + \dfrac{1}{2m_2}\right]}$ (86)

Since KE is conserved

$$\boxed{(p'_{1i})^2 = (p'_{1f})^2} \quad (87) \quad \Rightarrow \quad \boxed{p'_{1i} = \pm p'_{1f}} \quad (88)$$

The case $p'_{1i} = p'_{1f}$ is when the particles do not interact. The correct answer when they collide is

$$\boxed{p'_{1i} = -p'_{1f}} \quad (89)$$

and of course $\boxed{p'_{2i} = -p'_{1i} = +p'_{1f} = -p'_{2f}}$ (90)

$$\Rightarrow \quad \boxed{\begin{aligned} p'_{1f} &= -p'_{1i} \\ p'_{2f} &= -p'_{2i} \end{aligned}} \quad (91)$$

The particles reverse their momenta in the COM frame.

Really simple! Now to go back to the lab frame we use

$$\boxed{p'_{1f} = m_1 v'_{1f} = m_1 (v_{1f} - v_0) = p_{1f} - m_1 v_0} \quad (92)$$

$$\Rightarrow \quad \boxed{p_{1f} = p'_{1f} + m_1 v_0} \quad (93)$$

$$\boxed{p_{2f} = p'_{2f} + m_2 v_0} \quad (94)$$

So let's put it together

(95)

$$\boxed{p'_{1f} = -p'_{1i} = -m_1 v'_{1i} = -m_1(v_{1i} - v_0) = -p_{1i} + m_1 v_0}$$

$$\boxed{p_{1f} = p'_{1f} + m_1 v_0 = -p_{1i} + 2m_1 v_0} \quad (96)$$

and $\boxed{p_{2f} = -p_{2i} + 2m_2 v_0}$ (97)

Plug in for $v_0 = \dfrac{p_{1i} + p_{2i}}{m_1 + m_2}$

(98)

$$\boxed{\begin{aligned} p_{1f} &= -p_{1i} + \frac{2m_1}{m_1+m_2}(p_{1i}+p_{2i}) = \frac{m_1-m_2}{m_1+m_2}p_{1i} + \frac{2m_1 p_{2i}}{m_1+m_2} \\ p_{2f} &= -p_{2i} + \frac{2m_2}{m_1+m_2}(p_{1i}+p_{2i}) = \frac{m_2-m_1}{m_1+m_2}p_{2i} + \frac{2m_2 p_{1i}}{m_1+m_2} \end{aligned}}$$

Same as (66), (67) with a lot less effort!

We can re-express (95), (96) as

$\boxed{v'_{1f} = -v'_{1i}}$ (99) $\boxed{v'_{2f} = -v'_{2i}}$ (100)

The velocities in the COM frame just reverse after a perfectly elastic collision

Consider an example of how to use the COM frame.

Say $\boxed{m_1 = 5\,kg}$ (101) $\boxed{v_{1i} = 10\,m/s}$ (102) $\boxed{m_2 = 3\,kg}$ (103)

$\boxed{v_{2i} = -1\,m/s}$ (104)

(105)

$$\boxed{p_i = m_1 v_{1i} + m_2 v_{2i} = (50 - 3)\,kg\,m/s = 47\,kg\,m/s}$$

$$\Rightarrow \boxed{v_0 = \frac{p_i}{m_1 + m_2} = \frac{47\,kg\,m/s}{8\,kg} = 5.875\,m/s} \quad (106)$$

$$\boxed{\begin{aligned} v_{1i}' &= v_{1i} - v_0 = 4.125\,m/s \\ v_{2i}' &= v_{2i} - v_0 = -6.875\,m/s \end{aligned}} \quad (107)$$

We know that velocities reverse in the COM frame

$$\boxed{\begin{aligned} v_{1f}' &= -v_{1i}' = -4.125\,m/s \\ v_{2f}' &= -v_{2i}' = +6.875\,m/s \end{aligned}} \quad (108)$$

Now go back to the lab frame

(109)

$$\boxed{\begin{aligned} v_{1f} &= v_{1f}' + v_0 = -4.125\,\frac{m}{s} + 5.875\,m/s = 1.75\,m/s \\ v_{2f} &= v_{2f}' + v_0 = 6.875\,\frac{m}{s} + 5.875\,\frac{m}{s} = 12.75\,m/s \end{aligned}}$$

Check if the conservation laws are satisfied in the lab frame

(110)

$p_i = 47\,kg\,m/s$

$$\boxed{\begin{aligned} p_f &= 5\,kg \times 1.75\,\frac{m}{s} + 3\,kg \times 12.75\,m/s \\ &= 47\,kg\,m/s \end{aligned}}$$

$$K_i = \frac{1}{2} \times 5\,kg \times (10\,m/s)^2 + \frac{1}{2} \times 3\,kg \times (-1\,m/s)^2 \quad (111)$$

$$= 250\,J + 1.5\,J = 251.5\,J$$

$$K_f = \frac{1}{2} \times 5\,kg \times (1.75\,m/s)^2 + \frac{1}{2} \times 3\,kg \times (12.75\,m/s)^2 \quad (112)$$

$$= 7.65625\,J + 243.84375\,J = 251.5\,J$$