Linear Momentum

Momentum is another of those words that is used in common speech, which is also used in physics. To see why momentum is useful let us consider a system of two interacting particles. No forces external to the system are present.

So the force on 1 is due to 2 and the force on 2 is due to 1. Newton II says

$$\boxed{\vec{F}_{21} = m_1 \frac{d^2\vec{r}_1}{dt^2} = m_1 \frac{d\vec{v}_1}{dt} = \frac{d}{dt}(m_1\vec{v}_1)}$$

$$\boxed{\vec{F}_{12} = m_2 \frac{d^2\vec{r}_2}{dt^2} = m_2 \frac{d\vec{v}_2}{dt} = \frac{d}{dt}(m_2\vec{v}_2)}$$

But, by Newton's III Law

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}(m_1\vec{v}_1) = -\frac{d}{dt}(m_2\vec{v}_2)}$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = 0}$$

We have discovered a conservation law!

The combination $m\vec{v}$ is so useful that it is given a name, momentum.

$$\boxed{\text{Momentum} = \vec{p} = m\vec{v}}$$

So, in the above case of two particles

$$\boxed{\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0} \quad \text{or} \quad \boxed{\vec{p}_1 + \vec{p}_2 = \text{Constant.}}$$

So the total momentum of an isolated two-particle system is conserved.

Does this hold for more particles? let's try with 3. By Newton II

$$\boxed{\begin{aligned} \vec{F}_{21} + \vec{F}_{31} &= \frac{d}{dt}(m_1\vec{v}_1) = \frac{d\vec{p}_1}{dt} \\ \vec{F}_{12} + \vec{F}_{32} &= \frac{d\vec{p}_2}{dt} \\ \vec{F}_{13} + \vec{F}_{23} &= \frac{d\vec{p}_3}{dt} \end{aligned}}$$

Add all the eq^{n}s.

$$\boxed{\frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = \underbrace{\vec{F}_{21} + \vec{F}_{12}}_{0} + \underbrace{\vec{F}_{31} + \vec{F}_{13}}_{0} + \underbrace{\vec{F}_{23} + \vec{F}_{32}}_{0}}$$

$$\boxed{\frac{d}{dt}\left(\vec{p}_1+\vec{p}_2+\vec{p}_3\right)=0}$$

We can in fact show that this works for any isolated system of any number of particles.

The total momentum of an isolated system is conserved

Isolated means that no external force acts on the system. All the forces are produced by objects in the system.

What if the system is not isolated? Let $\alpha, \beta = 1, \ldots N$ be the different particles.

Let $\boxed{\vec{F}_{\alpha\beta} \text{ be the force due to } \alpha \text{ on } \beta}$

Let $\boxed{\vec{F}_{ext,\beta} \text{ be the total external force on } \beta}$

Then $$\boxed{\frac{d\vec{p}_\beta}{dt} = \sum_{\alpha\neq\beta} \vec{F}_{\alpha\beta} + \vec{F}_{ext,\beta}}$$

Add all the momenta

$$\boxed{\sum_{\beta=1}^{N} \frac{d\vec{p}_\beta}{dt} = \frac{d}{dt}\left(\sum_{\beta=1}^{N} \vec{p}_\beta\right) = \sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{N} \vec{F}_{\alpha\beta} + \sum_{\beta=1}^{N}\vec{F}_{ext,\beta}}$$

By Newton III the 1st sum on the RHS vanishes

So $$\boxed{\frac{d}{dt}\left(\sum_{\beta=1}^{N} \vec{p}_\beta\right) = \vec{F}_{tot,\,ext} = \sum_{\beta=1}^{N} \vec{F}_{ext,\beta}}$$

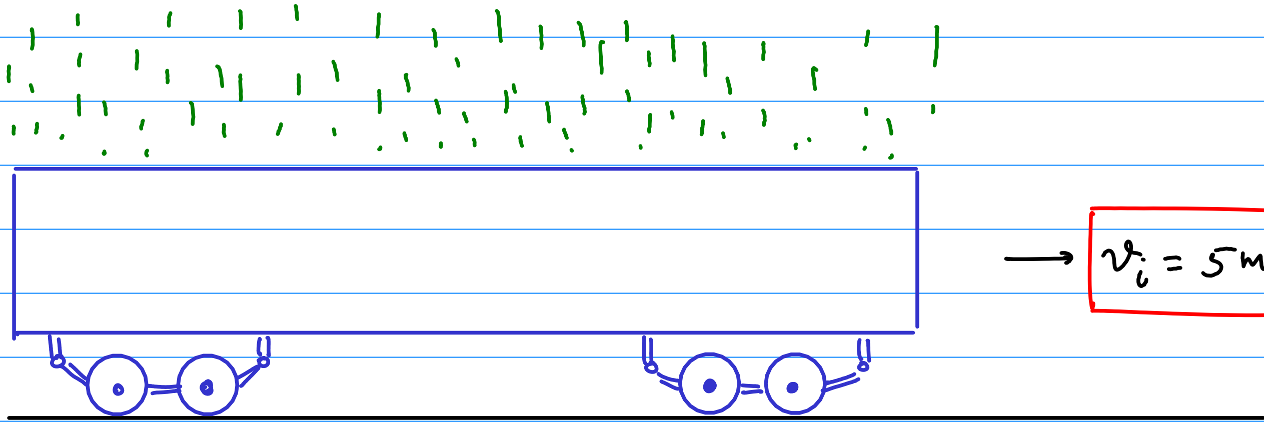
So, if there are external forces the total momentum of the system changes. The rate of change of total momentum is the total external force.

One very nice thing about momentum is that it is a vector, with 3 components.

Sometimes, it happens that there are external forces along some directions but not others. Along any direction in which $F_{ext} = 0$ the total momentum is conserved.

Example 1: An empty freight car ($m_c = 10{,}000\,kg$) is coasting along frictionless rails at $5\,m/s$.

Heavy rain is falling vertically down at the rate of $\boxed{50\,kg/s}$ into the car. What is its speed after $\boxed{100\,secs}$?



$\rightarrow$ $\boxed{v_i = 5\,m/s}$

The system consists of the freight car + raindrops. External vertical forces (gravity, normal force) are acting on the system but no horizontal external forces act on the system

$$\Rightarrow \boxed{p_{x,tot} = \text{constant.}}$$

$$\boxed{p_{x,i} = 10000\,\text{kg} \times 5\,\text{m/s} = 5\times10^4\ \text{kg}\cdot\text{m/s}}$$

$$\boxed{p_{x,f} = m_f v_{x,f}} \qquad \boxed{m_f = 10000 + 50\times100 = 15000\,\text{kg}}$$

$$p_{x,i} = p_{x,f} \Rightarrow 5\times10^4\ \text{kg}\cdot\text{m/s} = 15000\, v_{xf}$$

$$\Rightarrow \boxed{v_{xf} = 3.33\ \text{m/s}}$$

Example 2: Consider a head-on collision between a car ($m_1 = 2000\,\text{kg}$) moving East at $20\,\text{m/s}$ and a truck ($m_2 = 15{,}000\,\text{kg}$) moving West at $10\,\text{m/s}$. Initially ignore friction.

$$\boxed{v_{1i} = 20\,\text{m/s}} \qquad \boxed{v_{2i} = -10\,\text{m/s}}$$

$$\boxed{p_{1i} = 2000\,\text{kg} \times 20\,\text{m/s} = 40000\,\text{kg m/s}}$$

$$\boxed{p_{2i} = 15{,}000\,\text{kg} \times (-10\,\text{m/s}) = -150\,000\,\text{kg m/s}}$$

$$\boxed{p_{tot,i} = -110000 \text{ kg m/s}}$$

The two stick together after the collision. Since no forces act in the x-direction (remember we are ignoring friction)

$$\boxed{p_{tot,f} = (2000 \text{ kg} + 15000 \text{ kg})\, v_f = -110000 \text{ kg m/s}}$$

$$\Rightarrow \boxed{v_f = -6.5 \text{ m/s}}$$

Now lets assume that the collision takes $\boxed{0.05 \text{ sec}}$. The average acceleration of the car during the collision

$$\boxed{a_{av,car} = \frac{-6.5 \text{ m/s} - 20 \text{ m/s}}{0.05 \text{ sec}} = -530 \text{ m/s}^2 = 53\ g!!}$$

$$\boxed{a_{av,truck} = \frac{-6.5 \text{ m/s} - (-10 \text{ m/s})}{0.05 \text{ sec}} = 70 \text{ m/s}^2}$$

This lets us find the average collisional force acting on the car.

$$\boxed{F_{av,car} = m_{car}\, a_{av,car} = 2000 \text{ kg} \times (-530 \text{ m/s}^2) = -1.06 \times 10^6 \text{ N}}$$

of course, by Newton III $F_{av,Truck} = 1.06 \times 10^6$ N

The frictional forces are of order

$$\mu m g \simeq 0.1 m g$$

This would be roughly 15000 N for the truck and 2000 N for the car. Clearly, frictional forces are small compared to collisional forces.