Rocket Science

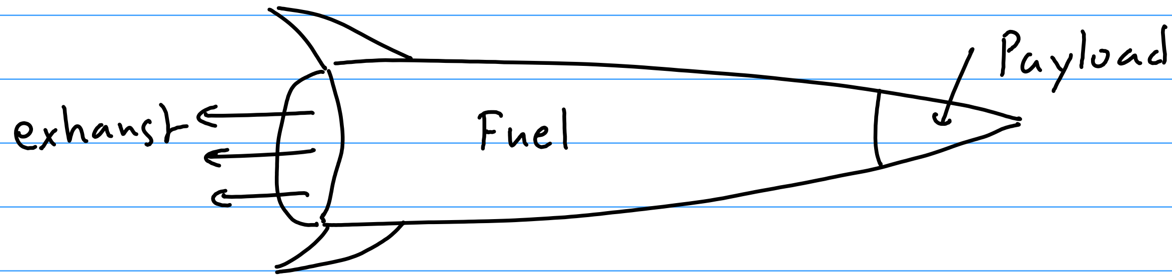
A rocket consists of a large tank of fuel and a tiny payload which has to be boosted either into orbit or into interplanetary space.

Let us first consider the simplest case. The rocket is far from any mass and there are no external forces on it.

Let $\boxed{M_F}$ 1 be the mass of the fuel, and $\boxed{M_P}$ 2 the mass of the payload. We will also need two other quantities, the burn rate $\boxed{\dot{m}}$ 3, which is the number of kg of fuel burned per sec

$$\boxed{[\dot{m}] = \frac{M}{T}}$$ 4 units kg/s

and the velocity of the exhaust gas relative to the rocket $\boxed{V_{ex}}$ 5



Since $\boxed{\vec{F}_{ext} = 0}$ 6 $\vec{p}$ is conserved.

Let us consider a time t after the rocket is switched on. The mass of the rocket at this time is

$$\boxed{M(t) = M_P + M_F - \dot{m}t} \quad 7$$

and its velocity is $\boxed{v(t)}$ 8. A time Δt later the mass is

$$\boxed{M(t) - \Delta m} \quad 9$$

and the speed is $\boxed{v(t) + \Delta v}$. 10

Δm has left the rocket as exhaust at a relative velocity of $-v_{ex}$ (going backwards). So the actual velocity of the exhaust in the lab frame is

$$\boxed{-v_{ex} + v(t)} \quad 11$$

12 $\boxed{P_i = M(t)\,v(t)}$; $\boxed{P_f = (M(t) - \Delta m)(v(t) + \Delta v) + \Delta m(-v_{ex} + v(t))}$ 13

Momentum conservation implies

$$M(t)\,v(t) = (M(t) - \Delta m)(v(t) + \Delta v) + \Delta m(-v_{ex} + v(t))$$

$$= M(t)v(t) - \cancel{\Delta m\, v(t)} + M(t)\Delta v - \Delta m \Delta v - \Delta m\, v_{ex} + \cancel{\Delta m\, v(t)}$$

or

$$\boxed{0 = M(t)\,\Delta v - \Delta m\, v_{ex} - \Delta m\, \Delta v} \quad 14$$

divide by Δt and take the limit $\Delta t \to 0$

The last term $\Delta m \Delta v$ becomes negligible

$$M(t)\frac{dv}{dt} - \frac{dm}{dt} v_{ex} = 0$$

$$\boxed{M(t)\frac{dv}{dt} = \dot{m}\, v_{ex}} \quad (15)$$

$$\boxed{\frac{dv}{dt} = \frac{\dot{m}\, v_{ex}}{M_p + M_F - \dot{m}t}} \quad (16)$$

This is the rocket differential eq^n, with $\dot{m}$ and v_{ex} constant.

Integrate both sides between 0 and T

$$\int_0^T dt\, \frac{dv}{dt} = \dot{m} v_{ex} \int_0^T \frac{dt}{M_p + M_F - \dot{m}t}$$

or

$$\boxed{v(T) - v(0) = v_{ex} \ln\left(\frac{M_p + M_F}{M_p + M_F - \dot{m}T}\right)} \quad (17)$$

Let us see what is the final velocity of the rocket. This will be when the entire fuel is exhausted

$$\boxed{\dot{m} T_f = M_F} \quad (18)$$

Assuming $v(0) = 0$

$$\boxed{v(T_f) = v_{ex} \ln\left(\frac{M_p + M_F}{M_p}\right)} \quad (19)$$

Normal fuels have $\boxed{v_{ex} \simeq 1000\,m/s}$. 20 Suppose one wants to achieve a final $\boxed{v(T_f) = 10\, v_{ex}}$ 21

Escape velocity from the Earth's gravitational field is $11\,km/s = 11000\ m/s$.

Then $$\boxed{10 = \ln \frac{M_P + M_F}{M_P}}$$ 22

or $$1 + \frac{M_F}{M_P} = e^{10} = 22026$$

$$\Rightarrow \boxed{M_F \approx 22025\ M_P}$$ 23

The mass of the fuel should be 22 thousand times the mass of the payload!!

So rocket designers try hard to make v_{ex} as large as possible. For cryogenic $O_2 + H_2$ a v_{ex} of about 2400 m/s has been achieved.

So, for this v_{ex}, to achieve escape velocity

24 $\boxed{v(T_f) = 1.1 \times 10^4}$ $\boxed{v_{ex} = 2.4 \times 10^3}$ 25

$$\frac{v(T_f)}{v_{ex}} = 4.6 \Rightarrow \boxed{\frac{M_P + M_F}{M_P} = e^{4.6} \simeq 100}$$ 26

Not undoable!

Now let's see how the rocket behaves near the Earth's surface. Now the F_{ext} is not zero. If the rocket is pointed vertically upward $F_{ext} = -M(t)\, g$

So, Eq (15) becomes

$$\boxed{M(t)\frac{dv(t)}{dt} = \dot{m}\, v_{ex} - M(t)\, g} \quad (27)$$

$$\Rightarrow \boxed{\frac{dv(t)}{dt} = \frac{\dot{m}\, v_{ex}}{M_P + M_F - \dot{m}\, t} - g} \quad (28)$$

$$\Rightarrow \boxed{v(T) = v(0) + v_{ex} \ln \frac{M_P + M_F}{M_P + M_F - \dot{m}\, T} - gT} \quad (29)$$

Now at $T_f = \frac{M_F}{\dot{m}}$ for $v(0) = 0$

$$\boxed{v(T_f) = v_{ex} \ln\left(\frac{M_P + M_F}{M_P}\right) - g\, T_F} \quad (30)$$

Say (31) $\boxed{M_P = 100\,kg}$ $\boxed{M_F = 10000\,kg}$ (32)

The initial acceleration (at $t \to 0$) of the rocket is $\boxed{a_i = \frac{\dot{m}\, v_{ex}}{M(0)}}$ (33)

This had better be at least g, otherwise the rocket will not leave the Earth's surface

Let's make $\boxed{\frac{\dot{m}\, v_{ex}}{M(0)} = 1.5g}$ 34

So for $\boxed{v_{ex} = 2400\,m/s}$ 35

36

$$\boxed{\dot{m} = \frac{1.5 \times 9.8\, m/s^2 \times 10100\, kg}{2400\, m/s} = 62\, kg/s}$$

$\Rightarrow$ $\boxed{T_f = 161 \text{ sec.}}$ 37

The final velocity is

$$\boxed{\begin{aligned} v(T_f) &= 2400\,\frac{m}{s}\,\ln\frac{10100}{100} - 9.8\, m/s^2 \times 161\, s \\ &= 11076\, m/s - 1578\, m/s \\ &= 9498\, m/s \end{aligned}}$$ 38

The 11076 is the speed it would have attained without gravity.

You can see that gravity makes a large difference.