Rotations III - Angular Momentum and Torque

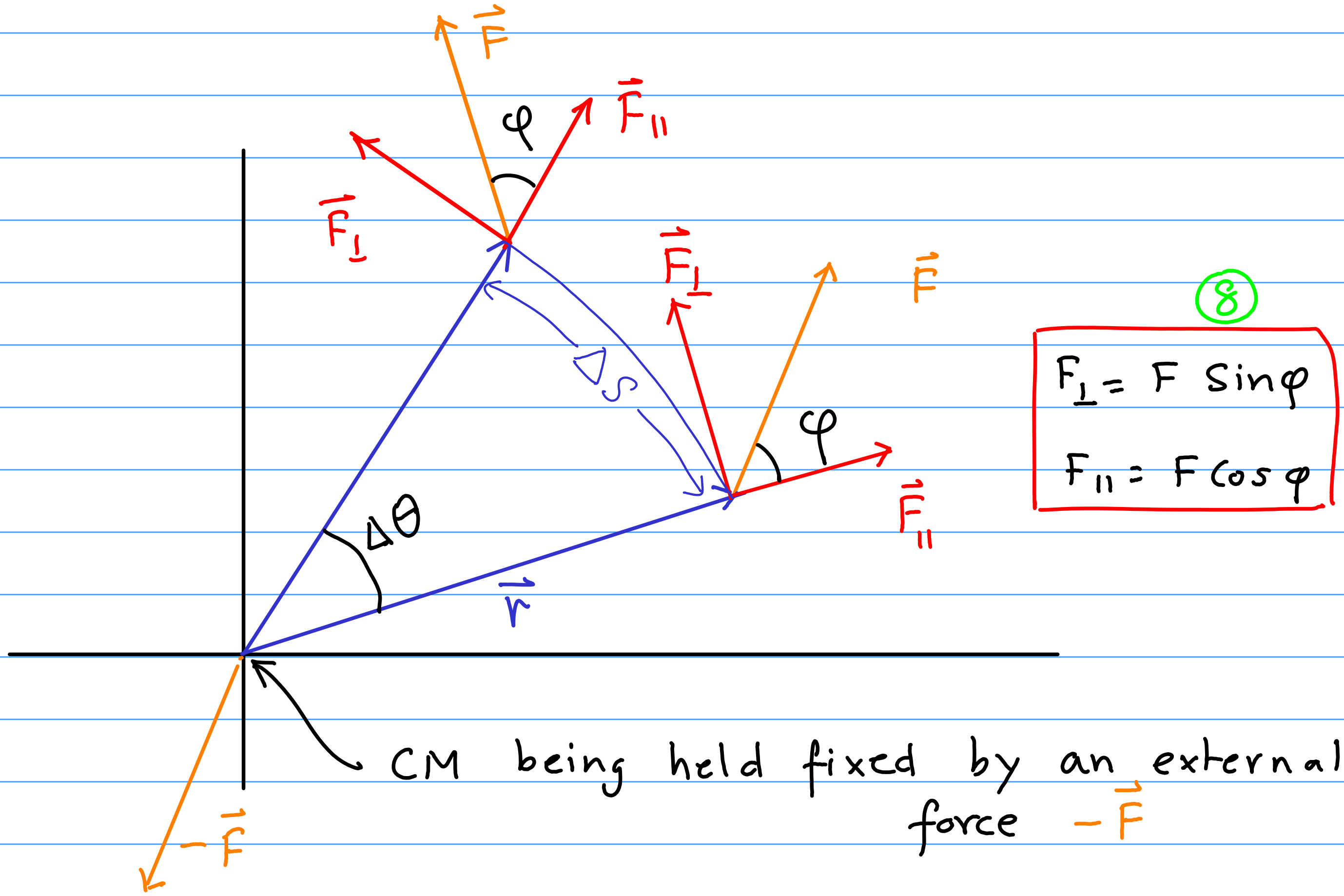
Let us compare linear and rotational motion in 1D. In the rotational case this corresponds to motion where the axis of rotation is fixed

Linear	Rotational
1a $x(t)$ = position	1b $\theta(t)$ = angular position
2a $v(t)$ = velocity = $\frac{dx}{dt}$	2b $\omega(t)$ = angular velocity = $\frac{d\theta}{dt}$
3a $a(t)$ = acceleration = $\frac{dv}{dt}$	3b $\alpha(t)$ = angular acc = $\frac{d\omega}{dt}$
4a $K = \frac{1}{2}Mv^2$	4b $K = \frac{1}{2}I\omega^2$

Now, there are other physical quantities we know in linear motion, such as force and momentum. Here are the corresponding angular quantities

5a p = momentum = Mv	5b L = Angular momentum = $I\omega$
6a F = force	6b τ = Torque
7a $F_{tot} = \frac{dp}{dt} = Ma$	7b $\tau_{tot} = \frac{dL}{dt} = I\alpha$

Let us first talk about torque. Consider a rigid body with the CM (the origin) fixed and rotating about a fixed axis, taken to be $\perp$ to the page



$$\boxed{\begin{aligned} F_{\perp} &= F \sin\varphi \\ F_{\parallel} &= F \cos\varphi \end{aligned}} \quad (8)$$

Now apply a force $\vec{F}$ at a point $\vec{r}$. Since the CM is being held fixed the CM acceleration must be zero

$$\boxed{\vec{R}_{CM} = \vec{v}_{CM} = \vec{a}_{CM} = 0} \quad (9)$$

So $$\boxed{\vec{F}_{ext, tot} = 0} \quad (10)$$

This means a force of $-\vec{F}$ is acting at the pivot point as shown.

It is clear that only $F_\perp$ is effective in turning the rigid body: If only $\bar{F}_\parallel$ is applied nothing will happen.

Now let us apply a constant magnitude F at a constant angle φ as the rigid body rotates through an angle $\Delta\theta$ as shown, between t and $t+\Delta t$

Let us assume there is no friction and apply the Work-Energy theorem

$$\boxed{W_{tot} = \Delta K = K(t+\Delta t) - K(t)} \quad (11)$$

The work done by $-\vec{F}$ at the pivot point is zero because the displacement is zero

$$\Rightarrow \boxed{W_{tot} = F_\perp \Delta S = F \sin\varphi \; r \, \Delta\theta} \quad (12)$$

$$K(t) = \frac{1}{2} I \big(\omega(t)\big)^2 \qquad K(t+\Delta t) = \frac{1}{2} I \big(\omega(t+\Delta t)\big)^2$$

$$\boxed{\begin{aligned}\Delta K &= \frac{1}{2} I \left[\big(\omega(t+\Delta t)\big)^2 - \big(\omega(t)\big)^2\right] \quad (13)\\ &= \frac{1}{2} I \left[\omega(t+\Delta t) - \omega(t)\right]\left[\omega(t+\Delta t) + \omega(t)\right] \approx I \omega(t) \Delta\omega\end{aligned}}$$

$$\boxed{r F \sin\varphi \; \Delta\theta \simeq I \, \omega(t) \, \Delta\omega} \quad (14)$$

$\Rightarrow$ divide by Δt and let $\Delta t \to 0$

$$\boxed{rF\sin\varphi\,\omega = I\omega\alpha} \quad (15)$$

or $$\boxed{rF\sin\varphi = I\alpha} \quad (16)$$

$$\boxed{rF\sin\varphi = rF_\perp = (\vec{r}\times\vec{F})\cdot\hat{k} = \tau = \text{Torque}} \quad (17)$$

Now we see the angular analogue of Newton's II law

$$\boxed{F_{tot} = ma \iff \tau_{tot} = I\alpha} \quad (18)$$

For a fixed axis $$\boxed{I\alpha = I\frac{d\omega}{dt} = \frac{d}{dt}(I\omega)} \quad (19)$$

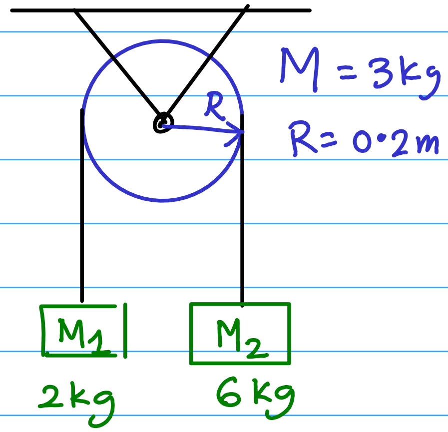
This makes it natural to define the angular momentum

$$\boxed{L = I\omega} \quad (20)$$

$$\boxed{\tau_{tot} = \frac{dL}{dt}} \quad (21)$$

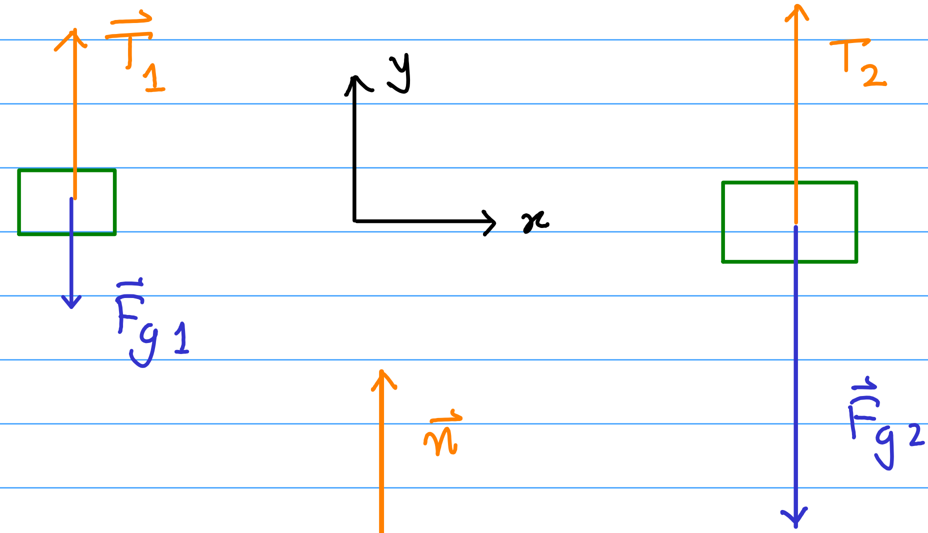
Let us do some examples. Start with

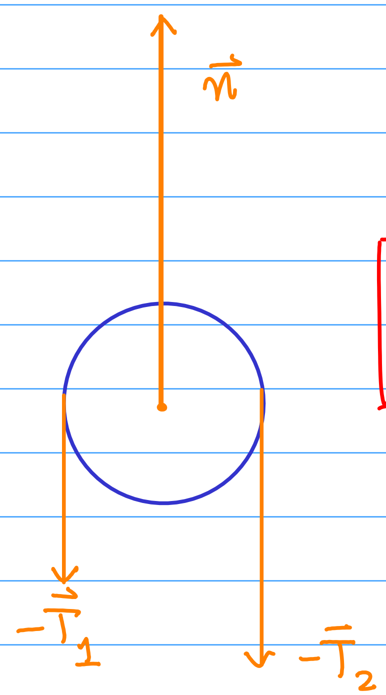
Example 1: Atwood's machine



Let the pulley have mass M, radius R and moment of inertia $\boxed{I = \frac{1}{2}MR^2}$ (solid cylinder) (22)

We have already solved this by energy conservation. Now we want to solve it using Newton's Laws. We are going to draw FBDs for the two masses and the pulley





23

$\vec{n}$ = Force of the ceiling on the axel

Since $a_{pulley} = 0$

$$\boxed{n = T_1 + T_2} \quad (24)$$

Clearly, since the rope is inextensible

$$\boxed{|\vec{a}_1| = |\vec{a}_2|} \quad (25) \qquad \boxed{\vec{a}_1 = +a\hat{j}} \quad (26) \qquad \boxed{\vec{a}_2 = -a\hat{j}} \quad (27)$$

Since the pulley "rolls" without slipping on the rope

$$\boxed{|\alpha| = \frac{a}{R}} \quad (28)$$ clockwise

or $$\boxed{\alpha = -\frac{a}{R}}$$ (29) (counterclockwise positive)

Let us apply Newton's laws.

Mass 1: $$\boxed{T_1 - M_1 g = M_1 a}$$ (30)

Mass 2: $$\boxed{T_2 - M_2 g = - M_2 a}$$ (31)

For the pulley we need the total torque

$-\vec{T}_1$ has a counterclockwise (positive) torque

$$\boxed{\tau_1 = T_1 R}$$ (32)

$-\vec{T}_2$ has a clockwise (negative) torque

$$\boxed{\tau_2 = - T_2 R}$$ (33)

(34)
$$\boxed{\begin{aligned}\tau_{tot} = (T_1 - T_2) R = I\alpha = -\frac{I a}{R} &= -\frac{1}{2} M R^2 \frac{a}{R} \\ &= -\frac{1}{2} M a R\end{aligned}}$$

or $$\boxed{(T_1 - T_2) = -\frac{1}{2} M a}$$ (35)

Collect all three eq^n^s

(36)
$$\boxed{\begin{aligned} T_1 - M_1 g &= M_1 a \\ -T_2 + M_2 g &= M_2 a \\ T_2 - T_1 &= \frac{1}{2} M a \end{aligned}}$$

add all three $\boxed{(M_2 - M_1)g = (M_1 + M_2 + \tfrac{1}{2}M)\,a}$ (37)

$$\boxed{\begin{aligned} a &= \frac{(M_2 - M_1)g}{M_1 + M_2 + \frac{1}{2}M} = \frac{(6-2)\times 9.8}{6+2+3/2} \\ &= 4.13\ m/s^2 \end{aligned}}$$ (38)

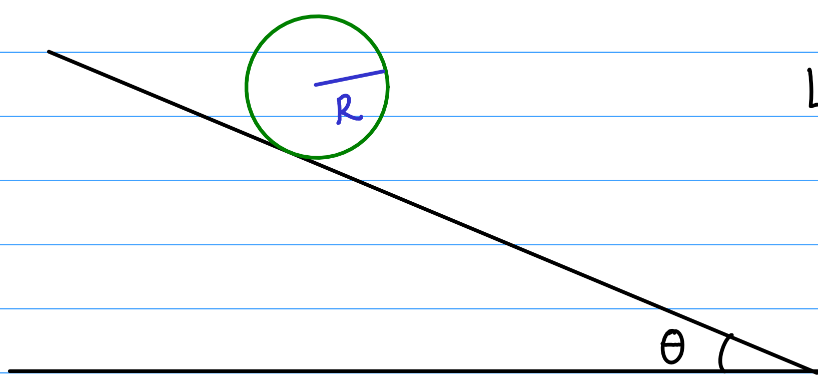
Let us find the tensions.

$$\boxed{T_1 = M_1 g + M_1 a = 2(9.8 + 4.13) = 27.9\ N}$$ (39)

$$\boxed{T_2 = M_2 g - M_2 a = 6(9.8 - 4.13) = 34\ N}$$ (40)

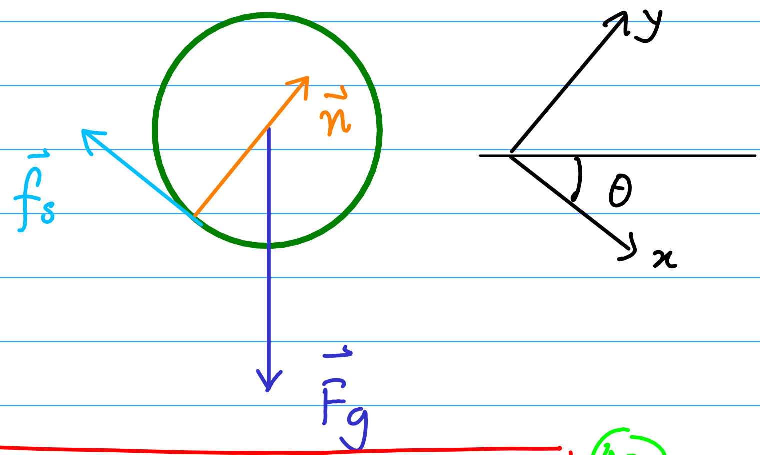
The main point is that $T_2 \neq T_1$. Some of the tension has been "used up" in accelerating the pulley angularly.

Example 2: Ball rolling down the incline



Let us draw a FBD of the ball of mass M, radius R and

$$\boxed{I = \frac{2}{5}MR^2}$$ (solid sphere) (41)



$$\boxed{\vec{F}_g = +Mg\sin\theta\,\hat{i} - Mg\cos\theta\,\hat{j}} \quad (42)$$

$$(43) \quad \boxed{\vec{n} = n\hat{j}} \qquad \boxed{\vec{f}_s = -f_s\,\hat{i}} \quad (44)$$

Now look 1st at the CM acceleration

$$\boxed{a_y = 0} \quad (45) \quad \Rightarrow \quad n - Mg\cos\theta = 0$$

$$\boxed{n = Mg\cos\theta} \quad (46)$$

$$\boxed{Mg\sin\theta - f_s = Ma_x} \quad (47)$$

Note that f_s is NOT EQUAL to $\mu_s n$ in general.

So we have two unknowns f_s, a_x. We need one more equation, which is the angular version of Newton's II Law.

Look at the torque around the CM of the ball. $\vec{F}_g$ and $\vec{n}$ go right through the CM, so

$$\boxed{\tau_g = \tau_n = 0} \quad (48)$$

$\vec{f}_s$ is $\perp$ to $\vec{r}$ at the point of contact,

So, because the torque due to f_s is clockwise

$$\boxed{\tau_s = -f_s R} \quad (49)$$

What is the angular acceleration α?

We know $\boxed{\omega = \frac{v_{CM}}{R}}$ (50) for rolling without slipping

$$\Rightarrow \quad \boxed{\alpha = \frac{d\omega}{dt} = \frac{a_{CM}}{R}} \quad (51)$$

Because ω is increasing clockwise

$$\boxed{\alpha = -\frac{a_x}{R}} \quad (52)$$

$$\boxed{\tau_{tot} = -f_s R = I\alpha = \frac{-I a_x}{R}} \quad (53)$$

$$\Rightarrow \quad f_s R = \frac{1}{R}\frac{2}{5} M R^2 a_x$$

$$\boxed{f_s = \frac{2}{5} M a_x} \quad (54)$$

Put this back in the linear eq^n

$$\boxed{Mg\sin\theta - \frac{2}{5} M a_x = M a_x} \quad (55)$$

$$g \sin\theta = \frac{7}{5} a_x$$

$$\boxed{a_x = \frac{5}{7} g \sin\theta} \quad 56$$

So $$\boxed{f_s = \frac{2}{5} M \frac{5}{7} g \sin\theta = \frac{2}{7} Mg \sin\theta} \quad 57$$

When will the ball start sliding rather than rolling down as the angle θ increases? The maximum angle for rolling is when

$$58$$

$$\boxed{f_s = f_{s,max} = \mu_s n = \mu_s Mg \cos\theta_{max}}$$

$$\Rightarrow \quad \boxed{\frac{2}{7} Mg \sin\theta_{max} = \mu_s Mg \cos\theta_{max}} \quad 59$$

$$\boxed{\tan\theta_{max} = \frac{7}{2}\mu_s} \quad 60$$

For $\boxed{\mu_s = 0.4}$ 61 $\quad \tan\theta_{max} = \frac{7}{5}$

$$\Rightarrow \quad \boxed{\begin{aligned}\theta_{max} &= 0.95 \text{ rad} \\ &= 54.5^\circ\end{aligned}} \quad 62$$