Rotational Motion

So far we have been focusing on the motion of objects considered as points. More precisely, we have been following the center of mass (CM) motion.

Recall

$$\boxed{\vec{R}_{CM} = \frac{\sum_\alpha M_\alpha \vec{r}_\alpha}{\sum_\alpha M_\alpha}} \quad (1)$$

or, if the mass is distributed continuously with density $\rho(\vec{r})$ (pronounced "rho of $\vec{r}$")

$$\boxed{M = \int d^3\vec{r}\ \rho(\vec{r})} \quad (2)$$

$$\boxed{\vec{R}_{CM} = \frac{\int d^3\vec{r}\ \vec{r}\,\rho(\vec{r})}{M}} \quad (3)$$

Also,

$$\boxed{\vec{v}_{CM} = \frac{d\vec{R}_{CM}}{dt}} \quad (4)$$

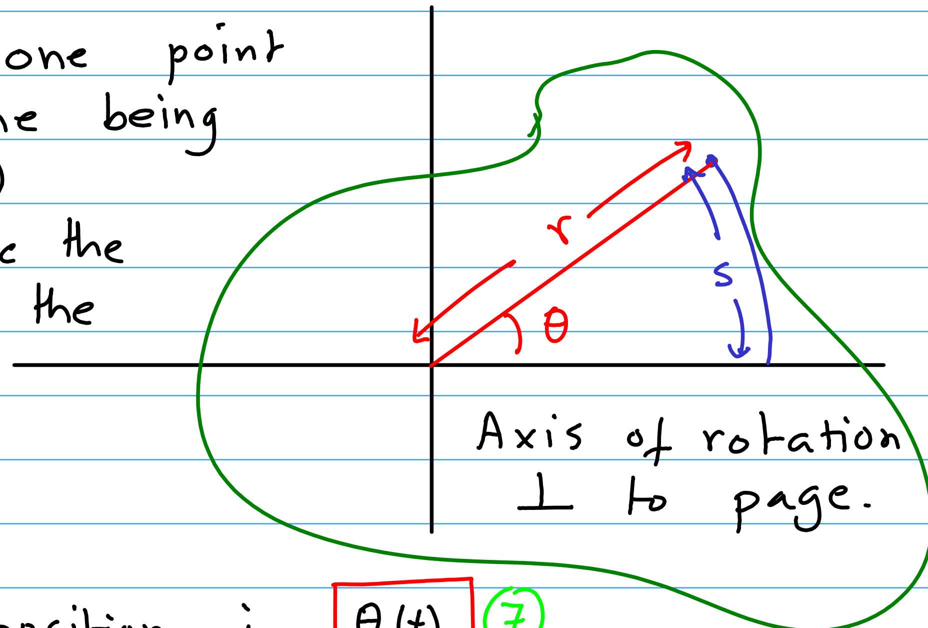
$$\boxed{M\vec{v}_{CM} = \vec{P}_{tot}} \quad (5)$$

$$\boxed{M\frac{d\vec{v}_{CM}}{dt} = M\vec{a}_{CM} = \vec{F}_{ext}} \quad (6)$$

Now let us turn to the other type of motion we commonly encounter, rotation about a fixed axis, with one point held fixed.

It is clear that for a rigid body all I need is one angle to describe the orientation.

I mark one point (not the one being held fixed) and measure the angle w.r.t. the x-direction.



Angular position is $\boxed{\theta(t)}$ (7)

Angular velocity $\boxed{\omega(t) = \frac{d\theta(t)}{dt}}$ (8)

Angular acceleration $\boxed{\alpha(t) = \frac{d\omega(t)}{dt}}$ (9)

If the point is a distance r away from the axis of rotation then it moves in a circle of fixed radius r. The arc $s(t)$ is

$\boxed{s(t) = r\,\theta(t)}$ (10) (definition of angle in radians)

So the tangential velocity of that point is

$$\boxed{v_t = \frac{ds}{dt} = r\omega} \quad (11)$$

Its tangential acceleration is

$$\boxed{a_t = \frac{dv_t}{dt} = r\alpha} \quad (12)$$

Because it moves in a circle of radius r with speed $v_t = r\omega$ it also has a centripetal acceleration which is radial

$$\boxed{a_r = -\frac{v_t^2}{r} = -\omega^2 r} \quad (13)$$

Example: Suppose I switch on a CD which has a button on it at $r = 0.05\,m$. The CD starts at $\omega_0 = 0$ and goes to $\omega_f = 100\,\frac{rads}{s}$ in 2 secs. If μ_s between the CD and the button is 0.3, when does the button begin to slip?

$$\boxed{\alpha = \frac{\Delta\omega}{\Delta t} = \frac{100 rads/s}{2s} = 50 rads/s^2} \quad (14)$$

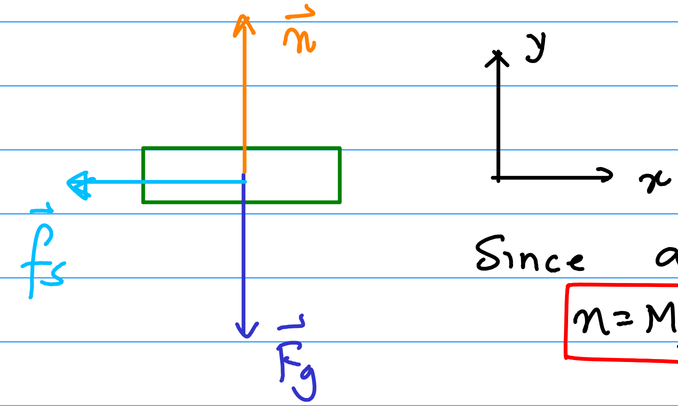
$$\boxed{\omega(t) = \alpha t} \quad (15) \qquad 0 \le t \le 2\,sec$$

$$\Rightarrow \quad \boxed{v_t(t) = r\omega(t) = r\alpha t} \quad (16)$$

(17)
$$\boxed{a_t(t) = r\alpha = 0.05\,m \times 50\,\frac{rad}{s^2} = 2.5\,m/s^2}$$

$$\boxed{a_r = -\frac{v_t^2}{r} = -r(\omega(t))^2 = -r\alpha^2 t^2} \quad (18)$$

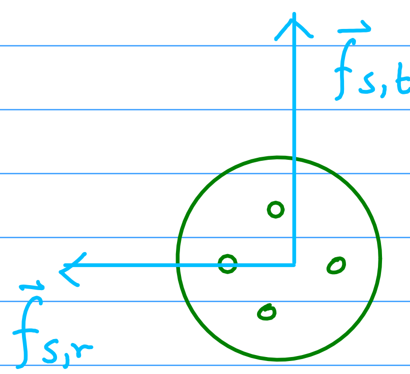
$$\boxed{|\vec{a}| = \sqrt{a_r^2 + a_t^2} = \sqrt{r^2\alpha^2 + r^2\alpha^4 t^4}} \quad (19)$$



Since $a_y = 0$

$$\boxed{n = Mg} \quad (20)$$

The FBD from the top view is



towards ← center of CD

(21)
$$\boxed{|\vec{f}_{s,max}| = \mu n = \mu Mg}$$

(22)
$$\text{So} \quad \boxed{|\vec{a}|_{max} = \mu g = 0.3g = 3\,m/s^2} \quad (g = 10\,m/s^2)$$

$$\text{So} \quad 3 = \sqrt{(2.5)^2 + r^2\alpha^4 t_{max}^4}$$

$$\sqrt{r^2\alpha^4 t_{max}^4} = \sqrt{3^2 - 2.5^2} = 1.66\,m/s^2$$

$$\text{or } r\alpha^2 t_{max}^2 = 1.66 \; m/s^2$$

$$t_{max}^2 = \frac{1.66}{0.05 \times 2500} = 0.0132 \; s^2$$

$$\boxed{t_{max} = 0.115 \text{ sec}} \tag{23}$$

Now let us consider the KE of a rotating object, which will lead us to an important concept, the moment of inertia of an object.

If the object consists of mass points M_α each of which is a distance d_α from the axis of rotation, then

$$\boxed{K = \sum_\alpha \frac{1}{2} M_\alpha v_\alpha^2 = \sum_\alpha \frac{1}{2} M_\alpha v_{\alpha,t}^2} \tag{24}$$

$$\boxed{K = \sum_\alpha \frac{1}{2} M_\alpha (d_\alpha \omega)^2 = \frac{1}{2}\left(\sum_\alpha M_\alpha d_\alpha^2\right)\omega^2} \tag{25}$$

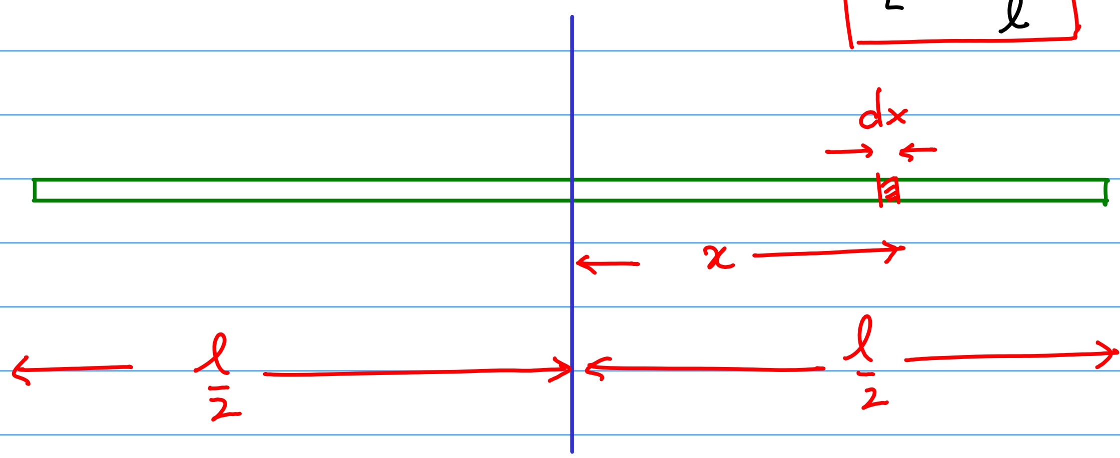
The quantity $\boxed{I = \frac{1}{2} \sum_\alpha M_\alpha d_\alpha^2}$ (26) is called the moment of inertia about the given axis. Units of I are $kg\, m^2$.

It is very important to realize that I does depend on the location of the axis of rotation, and its orientation.

Let us see how to calculate I for a few symmetric objects about various axes.

Example 1: A uniform, thin, rod of length ℓ and mass M.

The linear mass density $\boxed{\rho_L = \frac{M}{\ell}}$ (27)



1st let the axis pass through the geometric center, which is also the CM, and be $\perp$ to the rod.

Look at the small element of the rod marked in red. Its mass is

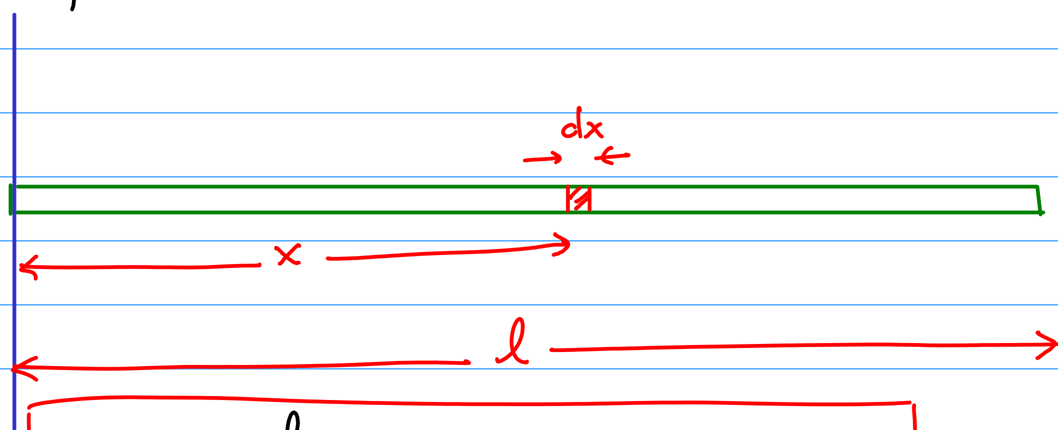
$$\boxed{dm = \rho_L \, dx = \frac{M}{\ell} dx} \quad (28)$$

and distance from the axis of rotation is x

So $$\boxed{I_{CM} = \int_{-\ell/2}^{\ell/2} \frac{M}{\ell} dx \; x^2 = \frac{M}{3\ell} \; x^3 \Big|_{-\ell/2}^{\ell/2}} \qquad (29)$$

$$\boxed{I_{CM} = \frac{1}{12} M\ell^3} \qquad (30)$$

How about if we move the axis to the end of the rod?



$$\boxed{I_{end} = \int_0^{\ell} \frac{M}{\ell} dx \; x^2 = \frac{1}{3} M\ell^2} \qquad (31)$$

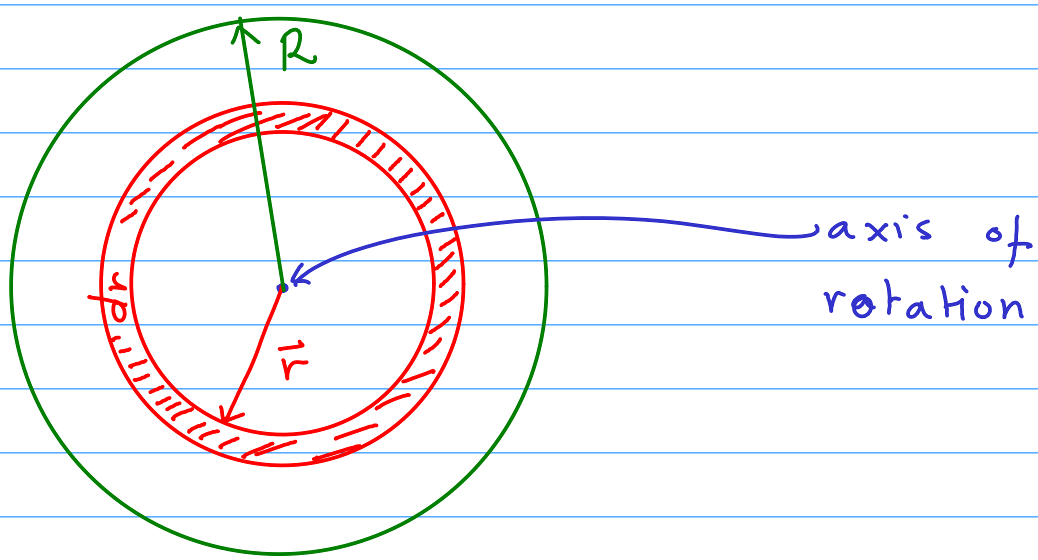
You can see the huge difference it makes!

The moment of inertia plays a role in rotational motion similar to the role the mass plays for linear motion but one should beware that I can change with the location and orientation of the axis.

Example 2: Take a thin circular disk of radius R and mass M.

Mass per unit area is $\boxed{\frac{M}{\pi R^2} = \rho_A}$ (32)

1st let us consider the axis which goes through the geometric center of the disk and is $\perp$ to the disk.



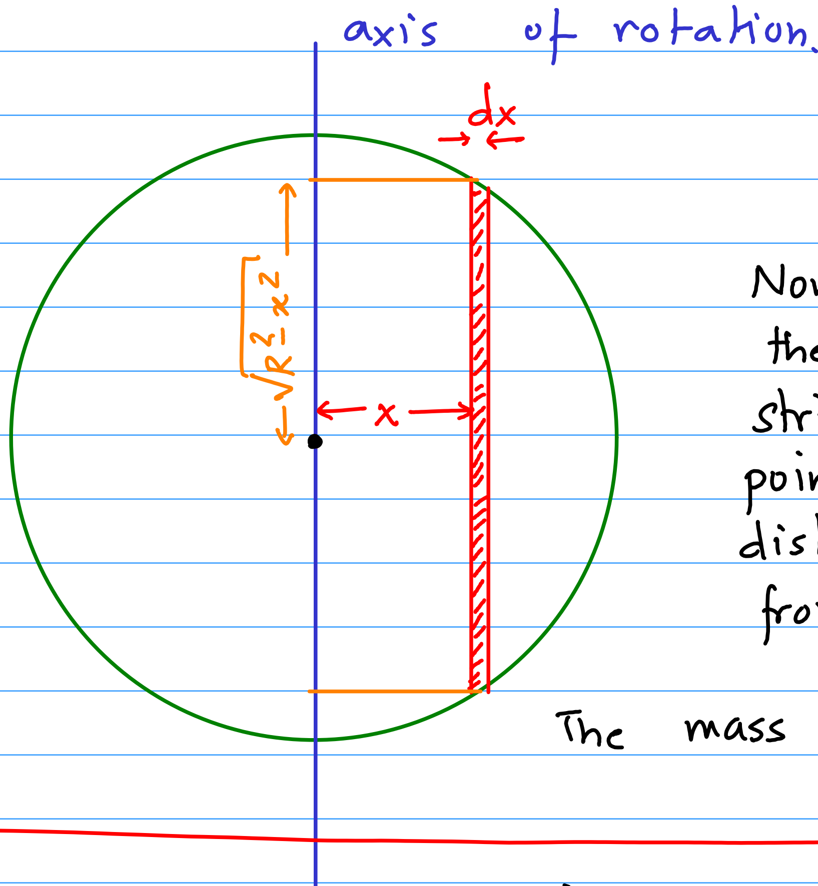
All the points in the red ring are at distance r from the axis of rotation. Their combined mass is

$$\boxed{dM = \rho_A \cdot (\text{Area of red ring})} \quad (33)$$

$$\boxed{dM = \frac{M}{\pi R^2}(2\pi r\,dr) = \frac{2M}{R^2} r\,dr} \quad (34)$$

So $\boxed{I_{CM,\perp} = \int_0^R \frac{2M}{R^2} r\,dr \; r^2 = \frac{2M}{R^2}\frac{1}{4}R^4 = \frac{MR^2}{2}}$ (35)

Now let us change the orientation of the axis of rotation so that it lies on the disk as shown.



Now choose the vertical red strip shown. All points are a distance x from the axis.

The mass of the strip is

$$\boxed{dM = \rho_A \,(\text{Area of strip}) = \frac{M}{\pi R^2}\, 2\sqrt{R^2-x^2}\,dx} \quad (36)$$

$$\Rightarrow \boxed{I_{CM,\parallel} = \int_{-R}^{R} \frac{2M}{\pi R^2}\sqrt{R^2-x^2}\,dx\; x^2} \quad (37)$$

Make the substitution $\boxed{x = R\sin\theta}$ (38)

(39) $\boxed{dx = R\cos\theta\, d\theta}$ $\boxed{\sqrt{R^2-x^2} = R\cos\theta}$ (40)

Limits : when $x = -R \quad \theta = -\frac{\pi}{2}$

$x = R \quad \theta = \frac{\pi}{2}$

$$I_{CM,\parallel} = \frac{2M}{\pi R^2} \int_{-\pi/2}^{\pi/2} R\cos\theta \; R\cos\theta \, d\theta \quad R^2 \sin^2\theta$$

$$= \boxed{\frac{2MR^2}{\pi} \int_{-\pi/2}^{\pi/2} \sin^2\theta \cos^2\theta \; d\theta} \qquad 41$$

Now $\sin^2\theta = \dfrac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\sin^2\theta \cos^2\theta = \frac{1}{4}\left(1-\cos^2(2\theta)\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{2}\left(1+\cos(4\theta)\right)\right)$$

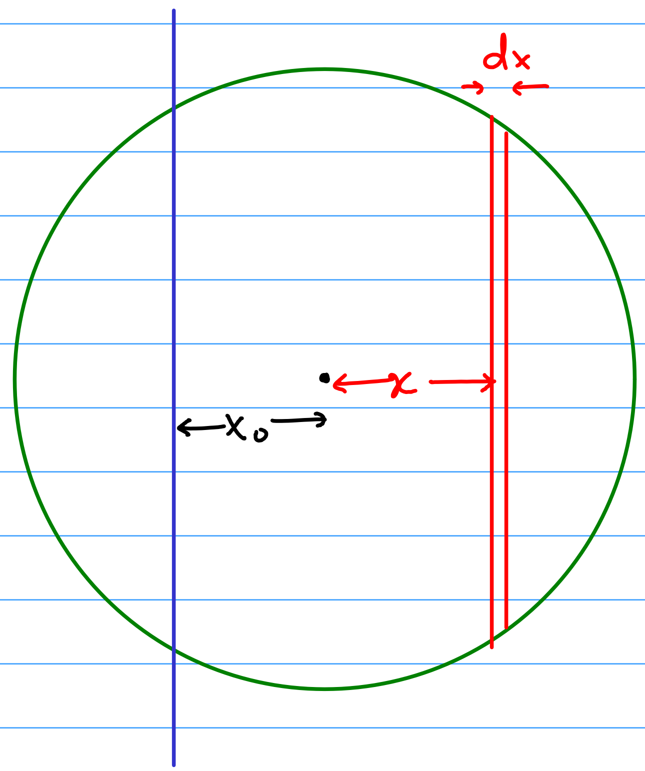
$$= \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$\int_{-\pi/2}^{\pi/2} d\theta \left[\frac{1}{8} - \frac{1}{8}\cos(4\theta)\right] = \frac{1}{8}\,\theta\Big|_{-\pi/2}^{\pi/2} - \frac{1}{32}\sin 4\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{8}$$

So for the thin disk $\boxed{I_{CM,\parallel} = \frac{MR^2}{4}}$ 42

Let us be a bit more ambitious. Suppose we shift the axis by some amount, leaving the orientation the same. For example, for the thin disk choose the axis of rotation in the plane of the disk, but offset by x_0 from the CM of the disk.



This seems hopelessly complicated, but is in fact very simple. Again choose the red strip. The distance from the new axis of rotation is

$$x + x_0$$

So $$\boxed{\int dm\,(x+x_0)^2 = \int dm\,(x^2 + x_0^2 + 2x\,x_0)} \quad (43)$$

Now $$\boxed{\int dm\, x^2 = I_{CM,\parallel} = \frac{MR^2}{4}} \quad (44)$$

$$\boxed{\int dm\, x_0^2 = x_0^2 \int dm = M x_0^2} \quad (45)$$

How about the cross term?

$$\boxed{\int dm \; 2x x_0 = 2x_0 \int dm \; x} \quad 46$$

Because the original axis went through the CM

$$\boxed{\int dm \, x = 0} \quad 47$$

For every dm with $x>0$ there is an equal dm with $x<0$

So

$$\boxed{I_{\parallel, x_0} = I_{CM,\parallel} + M x_0^2} \quad 48$$

This is the parallel axis theorem.

Let's go back to the rod and check.

For the axis going through the CM of the rod

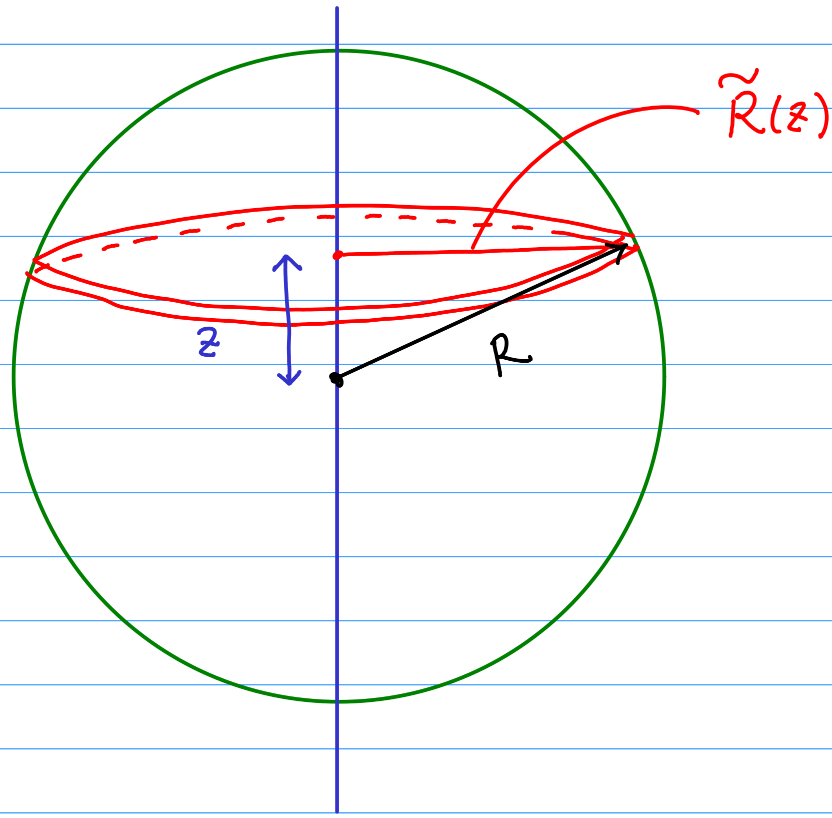
$$\boxed{I_{CM} = \frac{M\ell^2}{12}} \quad 49$$

When the axis is at the end $\boxed{x_0 = \ell/2}$ 50

$$\Rightarrow \boxed{I_{end} = I_{CM} + M\left(\frac{\ell}{2}\right)^2 = \frac{1}{12} M\ell^2 + \frac{1}{4} M\ell^2 = \frac{1}{3} M\ell^2}$$

which is correct. 51

Now consider a sphere, with the axis being through the CM



Cut the sphere into thin disks of thickness dz and radius $\boxed{\tilde{R}(z) = \sqrt{R^2 - z^2}}$ (52)

The mass of this thin disk is

$$\boxed{dM = \rho \text{ (Volume of thin disk)}} \quad (53)$$

$$\boxed{\rho = \text{Mass per unit volume} = \frac{M}{\frac{4}{3}\pi R^3}} \quad (54)$$

$$\boxed{\text{Volume of thin disk} = \pi \tilde{R}^2(z)\, dz = \pi dz (R^2 - z^2)} \quad (55)$$

$$\boxed{dM = \frac{M}{\frac{4}{3}\pi R^3} \pi dz (R^2 - z^2)} \quad (56)$$

We already know the moment of inertia for a disk $\frac{1}{2}MR^2$. So, for this thin disk

$$\boxed{dI = dM \, \frac{1}{2}\left(\tilde{R}(z)\right)^2 = \frac{M}{\frac{4\pi}{3}R^3}\pi\, dz\,(R^2-z^2)\,\frac{1}{2}(R^2-z^2)} \quad (57)$$

Now add the moments of inertia of all the disks that make up the sphere

$$\boxed{I_{CM}^{sphere} = \int_{-R}^{R} \frac{M}{\frac{4}{3}\pi R^3}\pi\, dz\, \frac{(R^2-z^2)^2}{2}} \quad (58)$$

$$= \frac{3}{8}\frac{M}{R^3}\int_{-R}^{R} dz\,(R^4 - 2R^2z^2 + z^4)$$

$$= \frac{3}{8}\frac{M}{R^3}\left\{R^4 z - 2R^2\frac{z^3}{3} + \frac{z^5}{5}\right\}\Bigg|_{-R}^{R}$$

$$= \frac{3}{8}\frac{M}{R^3}\left\{2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5\right\} = \frac{2}{5}MR^2$$

$$\boxed{I_{CM}^{sphere} = \frac{2}{5}MR^2} \quad (59)$$