Newton's Law of Universal Gravitation

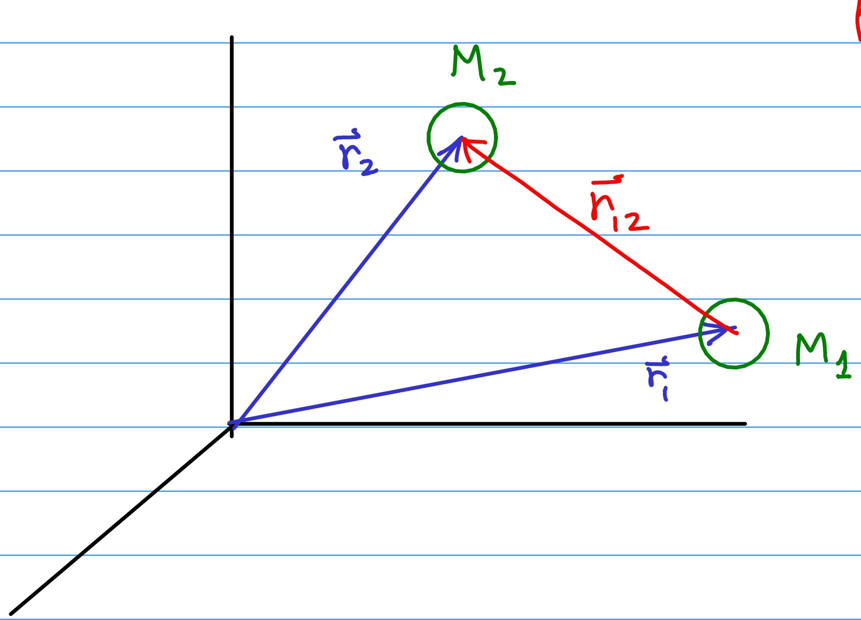
In 1666 Newton made the remarkable observation that the orbit of the Moon around the Earth and the fall of an apple at the Earth's surface resulted from the same phenomenon, called Universal Gravitation. The law is very simple to state in vector form

$$\boxed{\vec{F}_{12} = -\frac{G M_1 M_2}{r_{12}^2}\,\hat{r}_{12}} \quad (1)$$

$\vec{F}_{12}$ is the force due to body 1 (mass M_1) on body 2 (mass M_2).

$\vec{r}_{12}$ is the vector from 1 to 2

and $\hat{r}_{12}$ is the unit vector $\boxed{\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}}$ (2)



The force is attractive. That is the minus sign.

The force is proportional to the product of the masses. Note that the composition of the object does not matter. If the Moon were made of blue cheese it would still be attracted to the Earth by the same force, as long as its Mass remains the same.

The force is inverse square. This means it decreases as the square of the distance between the objects.

The constant G appearing in universal gravitation is called Newton's constant. In MKS units it value is

$$\boxed{G = 6.67 \times 10^{-11} \ N m^2/kg^2} \quad ③$$

As far as we can tell, any two objects anywhere in the universe attract each other with this law.

A characteristic of gravity is that it is very weak. Consider two baseballs, each of mass $\boxed{0.5 kg}$ ④ held $\boxed{5 cm = 5 \times 10^{-2} m}$ ⑤ apart. The attraction between them is

$$\boxed{|\vec{F}_{12}| = 6.67\times10^{-11}\ \frac{N\,m^2}{kg^2}\ \frac{0.5kg \times 0.5kg}{25\times10^{-4}\,m^2} \quad (6)}$$

$$\boxed{= 6.67\times10^{-9}\,N \quad \text{a really tiny force!}}$$

It takes something really large, such as the Earth, to create an appreciable gravitational force.

This law was verified in the lab by Henry Cavendish in 1797 by using a torsion balance, an extremely sensitive way to measure forces.

Using Newton's Universal Gravitation, we can estimate the mass of the Earth M_E. We know (by circumnavigating the globe) that the radius of the Earth is

$$\boxed{R_E = 6400\,km = 6.4\times10^{6}\,m} \quad (7)$$

So for a body of mass M, the gravitational force it feels at the Earth's surface is

$$\boxed{F = \frac{G M_E M}{R_E^2}} \quad (8)$$

But we know this to be Mg $\quad \boxed{g = 9.8\,m/s^2}$ (9)

So $$\frac{G M_E M}{R_E^2} = Mg \quad (10)$$

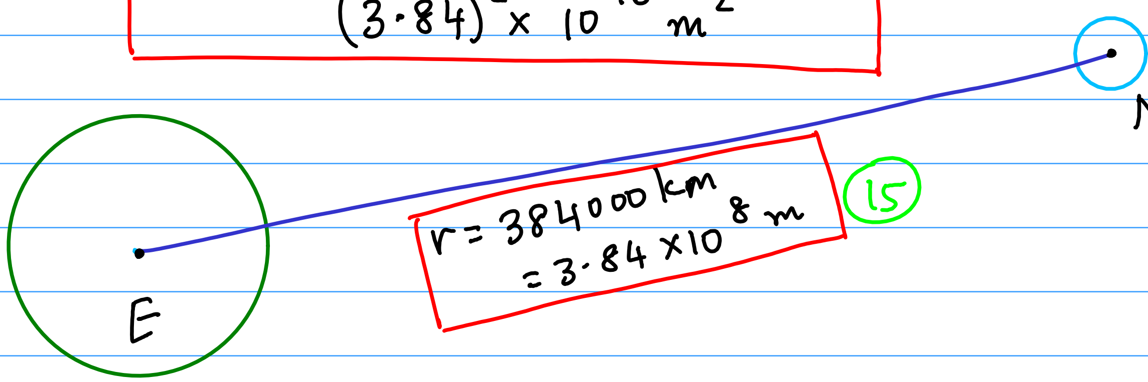
$$\text{or} \quad M_E = \frac{g R_E^2}{G} = \frac{9.8\ m/s^2 \times 6.4^2 \times 10^{12}\ m^2}{6.67 \times 10^{-11}\ N m^2/kg^2} \quad (11)$$

$$M_E = 6 \times 10^{24}\ kg \quad (12)$$

By the way, this explains why all masses accelerate due to gravity at the same rate g, because the force of gravity is proportional to the mass of the object.

How about the Moon? It is 384000 km (13) away (we know by bouncing lasers off its surface). If its mass is M_M (we don't need it) the force on it is

$$F_{EM} = \frac{6.67 \times 10^{-11}\ 6 \times 10^{24}\ M_M}{(3.84)^2 \times 10^{16}\ m^2} \quad (14)$$



$$F_{EM} = 2.71 \times 10^{-3}\ M_M \ \text{Newtons} \quad (16)$$

This pulls the Moon towards the Earth, and plays the role of the centripetal force

$$\boxed{F_{EM} = M_M a_c} \quad (17) \quad \Rightarrow \quad \boxed{a_c = 2.71 \times 10^{-3} \, m/s^2} \quad (18)$$

But we know that if the Moon is moving in a circle of radius r at speed v

$$\boxed{a_c = \frac{v^2}{r}} \quad (19)$$

$$\Rightarrow \quad \boxed{v = \sqrt{3.84 \times 10^8 \times 2.71 \times 10^{-3}} = 1.02 \times 10^3 \, m/s} \quad (20)$$

How long does the Moon take for one full orbit? This distance is $2\pi r$ and the speed is v

$$\Rightarrow \quad \boxed{T = \frac{2\pi r}{v} = \frac{2\pi \times 3.84 \times 10^8 \, m}{1.02 \times 10^3 \, m/s} = 2.37 \times 10^6 \, sec} \quad (21)$$

Recalling the $\boxed{1 \text{ day} = 24 \times 60 \times 60 = 86400 \text{ sec}}$ (22)

$$\boxed{T = \frac{2.37 \times 10^6}{86400} = 27.4 \text{ days.}} \quad (23)$$

The orbit of the Moon around the Earth is not a perfect circle. This orbit, and the orbits of planets around the Sun, are governed by Kepler's three laws

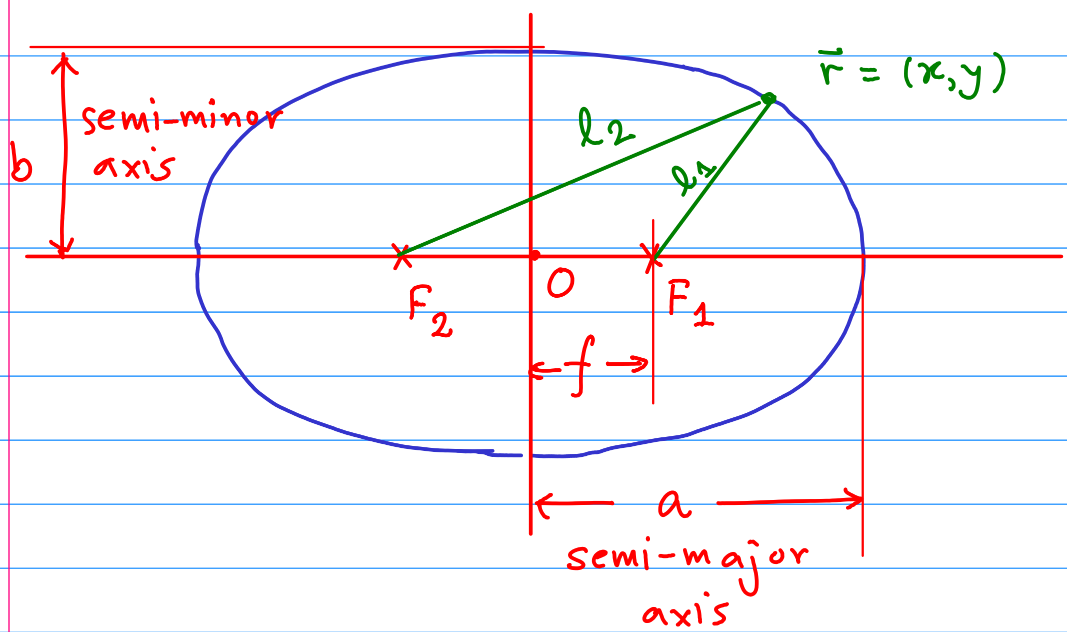
Kepler 1: The planets make ellipses with the Sun at one focus.

Kepler 2: The planets sweep out equal areas in equal times

Kepler 3: The time period of the orbit T and the semi-major axis "a" of the ellipse are related by

$$\boxed{T^2 \propto a^3} \quad (24)$$

Here is an ellipse



A point on the ellipse (x,y) satisfies

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad (25)$$

The ellipse has two foci (plural of focus) F_1 and F_2. The distance from the origin to the focus is

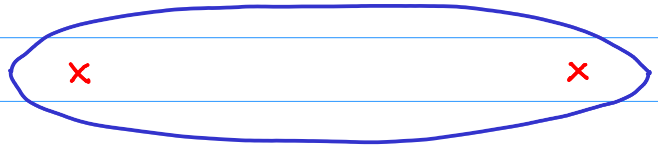
$$\boxed{f = a e} \quad (26)$$

where $\boxed{e = \text{eccentricity of the ellipse}}$ (27)

$$\boxed{0 \leq e < 1} \quad (28)$$

Also $\boxed{e = \sqrt{1 - b^2/a^2}}$ (29) or $\boxed{b = a\sqrt{1-e^2}}$ (30)

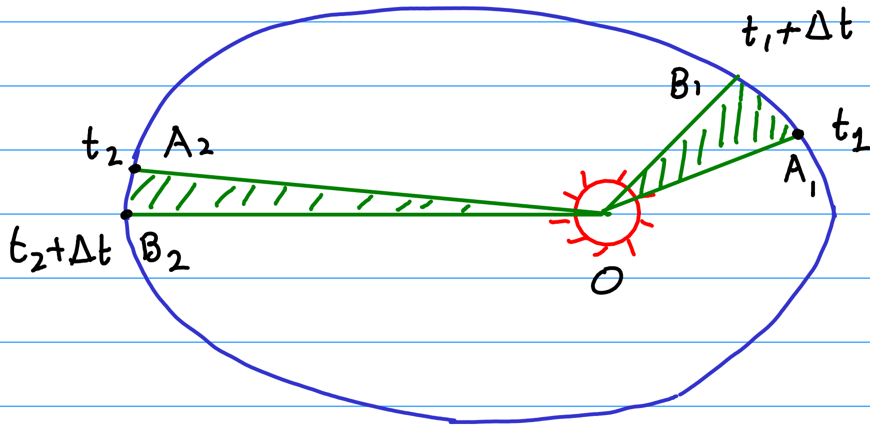
When $e=0$ the foci meet at O and $a=b$. The ellipse becomes a circle. As e increases the ellipse becomes more elliptical. Here is a case where e is close to 1



For Earth's orbit around the Sun $\boxed{e = 0.0167}$ (31)

For the Moon's orbit around Earth $\boxed{e = 0.055}$ (32)

Now we are ready to understand Kepler's laws. K1 just talks about the geometric shape of the orbit being an ellipse. To understand K2 we need to consider the motion in time. Let us consider t and $t + \Delta t$ at two different points on the orbit



K2 says that if we consider the area swept out by the planet in time Δt, which is the area OA_1B_1 at t_1 and OA_2B_2 at t_2 they are the same.

$$\boxed{\text{Area } OA_1B_1 = \text{Area } OA_2B_2} \quad (33)$$

We will see that this is related to the conservation of angular momentum with the Sun as the origin.

Finally, once we know the semi-major axis a, we can understand K3.

One can understand Kepler's third law most easily for a circular orbit. Let M_S = Mass of the Sun, and let M be the mass of the planet.

$M_S \gg M$ by several orders of magnitude.

Take the Sun as the origin and let the planet be in a circular orbit of radius r

$$\boxed{|\vec{F}_{sp}| = \frac{G M_s M}{r^2} = M a} \quad (34)$$

Since the force is purely radial and attractive, it plays the role of the centripetal force

$$\boxed{a = \frac{v^2}{r}} \quad (35)$$

$$\Rightarrow \quad \boxed{\frac{G M_s}{r^2} = \frac{v^2}{r}} \quad (36)$$

or $\quad v^2 = \dfrac{G M_s}{r}$ $\quad \boxed{v = \sqrt{\dfrac{G M_s}{r}}} \quad (37)$

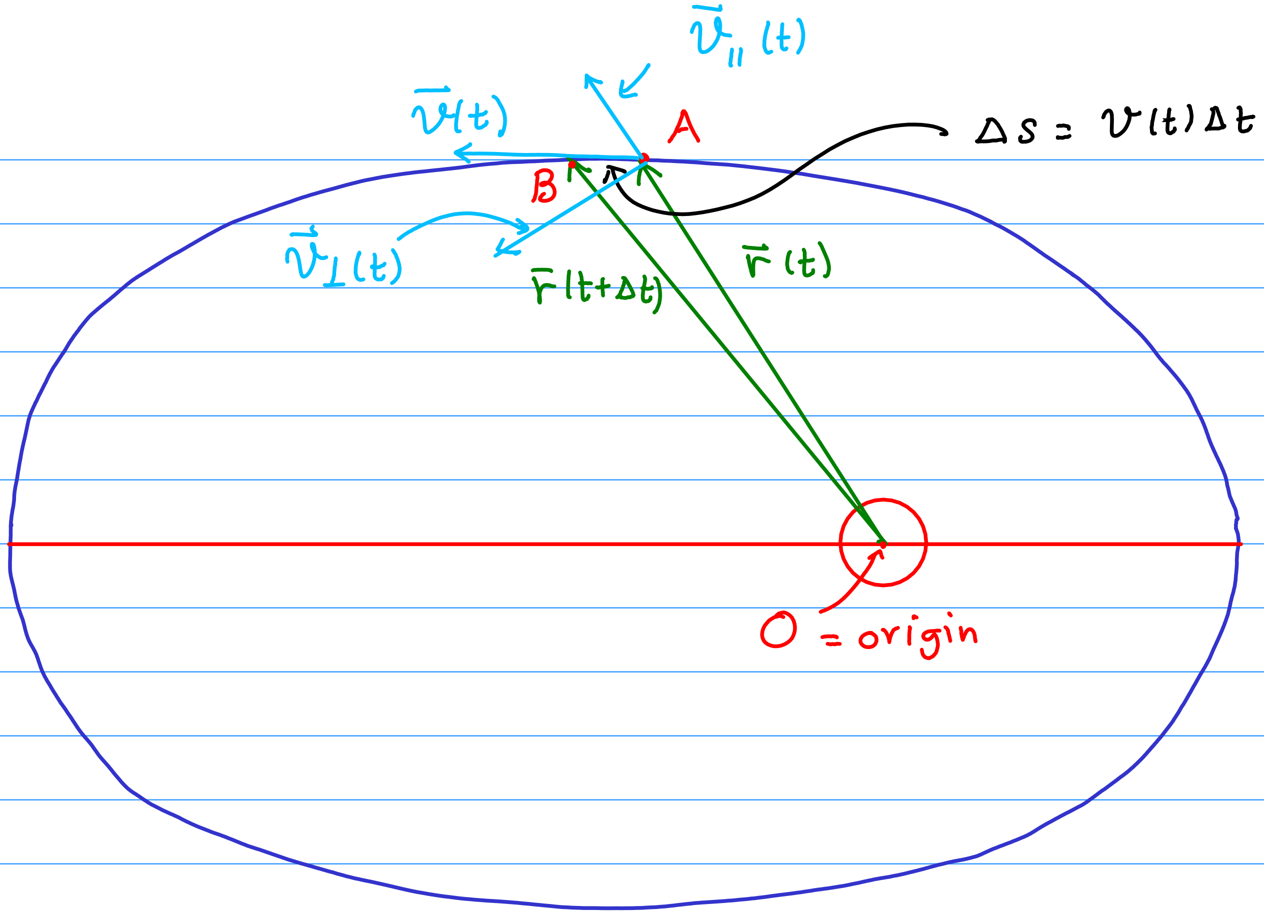
The time period is

$$\boxed{T = \frac{2\pi r}{v} = \frac{2\pi}{\sqrt{G M_s}} r^{3/2}} \quad (38)$$

or $$\boxed{T^2 = \frac{4\pi^2}{G M_s} r^3} \quad (39)$$

This is K3 specialized to a circular orbit.

Now let's go back to K2 and understand it as the consequence of the conservation of angular momentum



Between t and $t+\Delta t$ the planet travels from A to B or from $\vec{r}(t)$ to $\vec{r}(t+\Delta t)$

The distance travelled is

$$\boxed{AB = \Delta s = v(t)\,\Delta t} \quad (40)$$

The area of triangle OAB

$$= \frac{1}{2}\,\text{base} \times \text{height}$$

$$= \frac{1}{2}\,OA\,(v_{\perp}\,\Delta t)$$

$$\boxed{= \frac{1}{2}\left|\vec{r}(t) \times \vec{v}(t)\right|\,\Delta t} \quad (41)$$

Now the force is always directed towards the origin, so

$$\boxed{\bar{\tau} = \bar{r} \times \vec{F} = 0} \quad (42)$$

$$\Rightarrow \quad \boxed{\vec{L} = \text{conserved.}} \quad (43)$$

But $\boxed{\vec{L} = M\, \bar{r} \times \vec{v}} \quad (44)$

$\Rightarrow$ $\bar{r} \times \bar{v}$ is conserved. This is the reason behind K2.