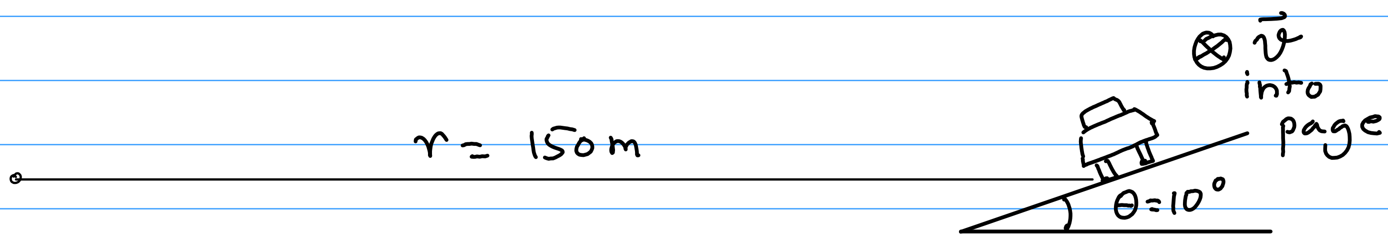
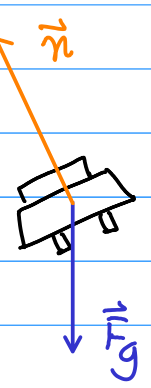
Banked curve with friction

A banked curve has a radius of 150m and a banking angle of $10°$. (i) Find the rated speed v_0 of the curve. (ii) If $\mu_s = 0.7$ find the maximum speed v_{max} at which the curve can be taken without skidding. (iii) If a car is travelling at $v = \frac{1}{2}(v_0 + v_{max})$ and the driver sees a deer, what is the maximum deceleration the car can have without skidding?

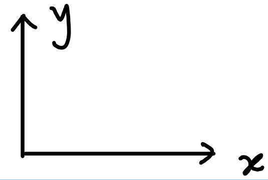


(i) Assume the car travels at the rated speed v_0. Then friction is not needed. Let us draw the FBD for this case



Since the car is travelling in a circle at constant speed the acceleration is $\boxed{v_0^2/r}$ 1 towards the center.

Choose coordinates



$$\boxed{\vec{a} = -\frac{v_0^2}{r}\hat{i}} \quad (2)$$

$$\boxed{\vec{F}_g = -Mg\hat{j}} \quad (3)$$

$$\boxed{\vec{n} = -n\sin\theta\,\hat{i} + n\cos\theta\,\hat{j}} \quad (4)$$

$$\boxed{\vec{F}_{tot} = -n\sin\theta\,\hat{i} + \hat{j}\,(n\cos\theta - Mg)} \quad (5)$$

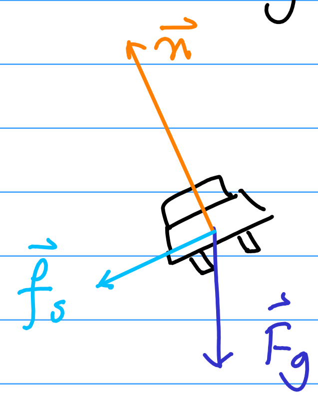
$$\boxed{a_y = 0 \Rightarrow F_{tot,y} = 0} \quad (6) \quad \Rightarrow \quad \boxed{n = \frac{Mg}{\cos\theta}} \quad (7)$$

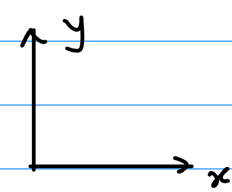
$$\boxed{F_{tot,x} = -n\sin\theta = -Mg\tan\theta = -M\frac{v_0^2}{r}} \quad (8)$$

$$\Rightarrow \boxed{v_0 = \sqrt{rg\tan\theta} = \sqrt{150\,m \times 9.8\,m/s^2 \times \tan 10^\circ} = 16.1\,m/s} \quad (9)$$

(ii) Now we want to find v_{max}. The frictional force will point towards the inside of the curve along the slope.

FBD





$$\boxed{\vec{F}_g = -Mg\,\hat{j}} \quad (10)$$

$$\boxed{\begin{aligned} \vec{n} &= -n\sin\theta\,\hat{i} + n\cos\theta\,\hat{j} \\ \vec{f}_s &= -f_s\cos\theta\,\hat{i} - f_s\sin\theta\,\hat{j} \end{aligned}} \quad (11)$$

$$\boxed{\vec{F}_{tot} = \hat{i}\,(-n\sin\theta - f_s\cos\theta) + \hat{j}\,(n\cos\theta - f_s\sin\theta - Mg)}$$ (12)

$$\boxed{a_y = 0 \Rightarrow \quad n\cos\theta - f_s\sin\theta = Mg}$$ (13)

$$\boxed{a_x = -\frac{v^2}{r} \Rightarrow \quad n\sin\theta + f_s\cos\theta = \frac{Mv^2}{r}}$$ (14)

Solve the simultaneous eqns. To find n, multiply the 1st eqn by $\cos\theta$, the second by $\sin\theta$ and add

$$\boxed{n = Mg\cos\theta + \frac{Mv^2}{r}\sin\theta}$$ (15)

To find f_s, multiply the 1st eqn by $-\sin\theta$, the second by $\cos\theta$ and add

$$\boxed{f_s = -Mg\sin\theta + \frac{Mv^2}{r}\cos\theta}$$ (16)

At the maximum speed v_{max} the car is about to skid

$$\Rightarrow \quad \boxed{f_s \equiv f_{s,max} = \mu_s n}$$ (17)

$$\boxed{-Mg\sin\theta + \frac{Mv_{max}^2}{r}\cos\theta = \mu_s\left(Mg\cos\theta + \frac{Mv_{max}^2}{r}\sin\theta\right)}$$ (18)

$$\boxed{\frac{v_{max}^2}{r}\left[\cos\theta - \mu_s\sin\theta\right] = g\left[\mu_s\cos\theta + \sin\theta\right]}$$ (19)

$$\boxed{\cos 10^\circ = 0.9848 \qquad \sin 10^\circ = 0.1736}$$ (20)

$$v_{max}^2 = rg\frac{[\mu_s \cos\theta + \sin\theta]}{[\cos\theta - \mu_s \sin\theta]} = rg\left[\frac{0.7\times 0.9848 + 0.1736}{0.9848 - 0.7\times 0.1736}\right]$$

$$= 0.999\, rg = rg \quad (21)$$

$$\Rightarrow \quad v_{max} = \sqrt{150\,m \times 9.8\,m/s^2} = 38.3\,m/s \quad (22)$$

(iii) Now the car travels at (23)

$$v = \frac{v_0 + v_{max}}{2} = \frac{16.1 + 38.3}{2} = 27.2\,m/s$$

$$\frac{v^2}{r} = 4.93\,m/s^2 \quad (24)$$

$$n = M\left[g\cos\theta + \frac{v^2}{r}\sin\theta\right] = 10.5\,\frac{m}{s^2}\,M$$

$$f_s = M\left[-g\sin\theta + \frac{v^2}{r}\cos\theta\right] = 3.15\,\frac{m}{s^2}\,M \quad (25)$$

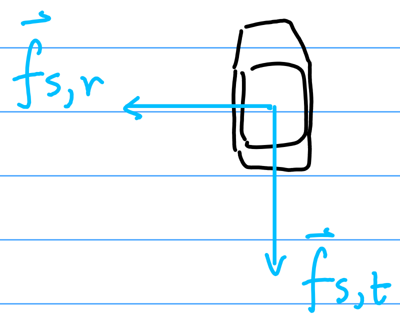
Now the maximum force of static friction at this n is

$$f_{s,max} = \mu_s n = 0.7\times 10.5\,M$$

$$= 7.35\,\frac{m}{s^2}\,M \quad (26)$$

When the car decelerates, in addition to the radial (centripetal) acceleration there

is also a tangential acceleration. This can only arise from static friction between the tyres and the road surface.
The FBD of the decelerating car as seen from above is



We already know $f_{s,r} = 3.15\,\frac{m}{s^2}\,M$

$$\boxed{f_{s,t} = M a_t} \quad (27)$$

a_t = tangential acceleration

The total force of friction is

$$\boxed{\vec{f}_{s,tot} = \vec{f}_{s,r} + \vec{f}_{s,t}} \quad (28)$$

$$\boxed{f_{s,tot} = \sqrt{f_{s,r}^2 + f_{s,t}^2} = M\sqrt{\left(\frac{v^2}{r}\right)^2 + a_t^2}} \quad (29)$$

When the car is about to skid

$$\boxed{f_{s,tot} = f_{s,max} = 7.35\,m/s^2\,M} \quad (30)$$

$$\Rightarrow \quad (7.35)^2 = (3.15)^2 + a_t^2$$

$$a_t^2 = 44.1\,(m/s^2)^2 \quad \Rightarrow \quad \boxed{a_t = 6.64\,m/s^2} \quad (31)$$

This is the maximum tangential deceleration.