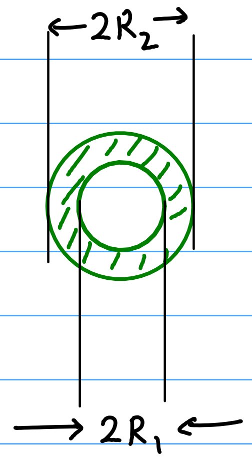
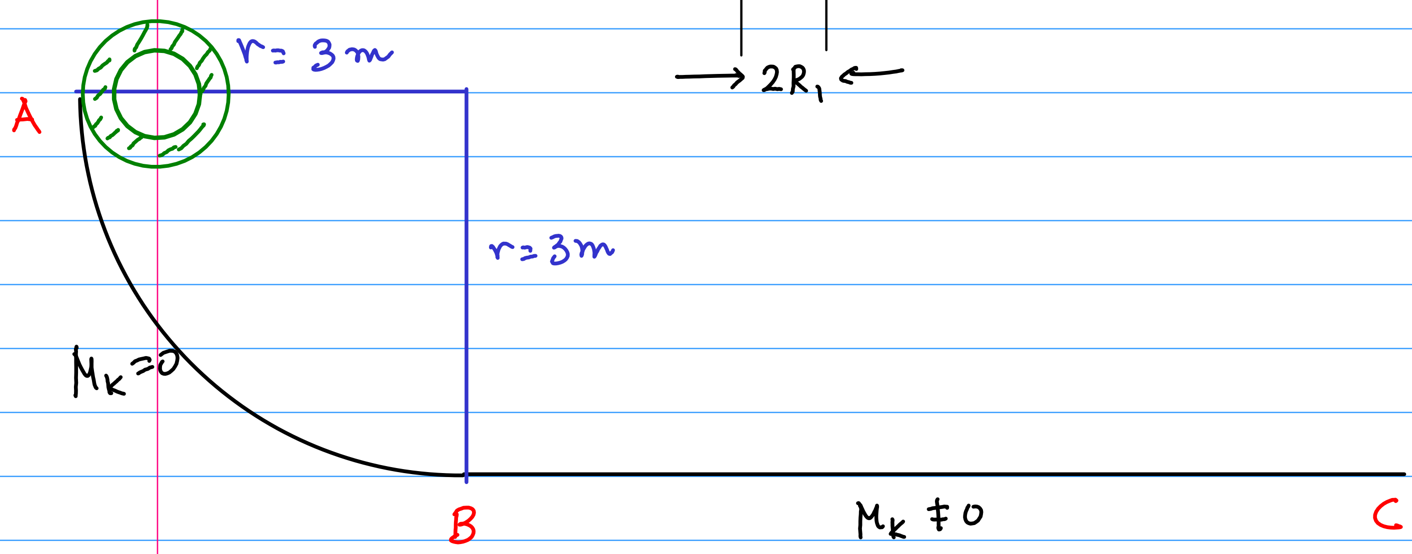
Potential Midterm Problems on Angular motion, rolling w/o slipping, torque, etc

Example 1: A hollow cylinder of mass $M = 10\ kg$ (1), inner radius $R_1 = 0.2\ m$ (2) and outer radius $R_2 = 0.3\ m$ (3) is at the top of the frictionless circular track shown. It slides down (without rotating) until it reaches the horizontal surface, where there is friction. It skids for a while until it starts rolling w/o slipping





1a: Find the moment of inertia of the hollow cylinder about its CM.

Consider the hollow cylinder as the superposition of a positive mass solid cylinder of radius R_2 and Mass $\boxed{M_2 = \rho_A \pi R_2^2}$ 4 and a negative mass solid cylinder of radius R_1 and Mass $\boxed{M_1 = -\rho_A \pi R_1^2}$ 5

$$\boxed{I_{tot} = \frac{1}{2} M_1 R_1^2 + \frac{1}{2} M_2 R_2^2 = \frac{1}{2} \pi \rho_A (R_2^4 - R_1^4)} \quad 6$$

But
$$\boxed{\rho_A = \frac{\text{Mass}}{\text{cross-sectional area}} = \frac{M}{\pi (R_2^2 - R_1^2)}} \quad 7$$

$$\boxed{I_{tot} = \frac{1}{2} M \frac{(R_2^4 - R_1^4)}{R_2^2 - R_1^2} = \frac{1}{2} M (R_1^2 + R_2^2)} \quad 8$$

$$= \frac{1}{2} 10\ \text{kg} \left((0.3\text{m})^2 + (0.2\text{m})^2 \right) = \boxed{0.65\ \text{kg m}^2} \quad 9$$

1b What is the speed of the cylinder when it reaches the bottom of the circular track?

Since $\boxed{f_k = 0}$ 10 $\boxed{W_{nc} = 0}$ 11 and $\boxed{\Delta E_{mech} = 0}$ 12

$$\Rightarrow \boxed{E_{mech,A} = E_{mech,B}} \quad 13$$

Choose the initial height of the CM to be $\boxed{h_A = 0}$ 14

$$\boxed{h_B = -(r - R_2) = -2.7m}$$ (15)

So
$$\boxed{\begin{aligned} E_{mech,A} &= K_A + U_{gA} = 0 \\ E_{mech,B} &= K_B + U_{gB} \end{aligned}}$$ (16)

Since the cylinder is not rotating

$$\boxed{K_B = \frac{1}{2} M v_B^2}$$ (17)

$$\boxed{U_{gB} = -Mg\,(r - R_2)}$$ (18)

$$\Rightarrow \quad 0 = \frac{1}{2} M v_B^2 - Mg\,(r - R_2)$$

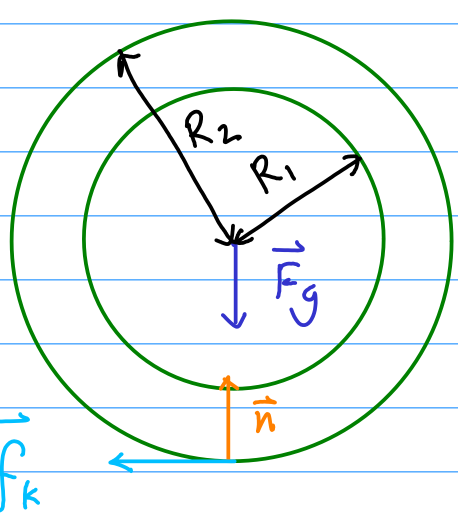
$$v_B = \sqrt{2g(r - R_2)} = \sqrt{2 \times 9.8 \frac{m}{s^2} \times 2.7m}$$

$$\boxed{v_B = 7.27\ m/s}$$ (19)

1c : Using the Impulse-Momentum thm in its linear and angular forms find the CM velocity of the cylinder when it finally rolls w/o slipping

FBD (21)

$$\boxed{\vec{F}_{tot} = -f_k \hat{i}}$$



(20)

$$\boxed{\begin{aligned} a_y &= 0 \\ \Rightarrow \quad n &= Mg \end{aligned}}$$

The only torque about the CM is due to f_k. Because it is clockwise

$$\boxed{\tau_{f_k} = -f_k R_2} \quad (22)$$

The impulse-momentum theorem says

$$\boxed{\vec{\mathcal{J}}_{tot} = \int_{t_i}^{t_f} \vec{F}_{tot}\, dt = \Delta \vec{p} = \vec{p}_f - \vec{p}_i} \quad (23)$$

CM momentum is always in the x-direction on the horizontal plane

$$\boxed{\mathcal{J}_{tot,x} = \int_{t_i}^{t_f} (-f_k)\, dt = M(v_c - v_B)} \quad (24)$$

Angular Impulse-momentum theorem says

$$\boxed{\mathcal{J}_{ang,tot} = \int_{t_i}^{t_f} \tau_{tot}\, dt = \Delta L = I(\omega_f - \omega_i)} \quad (25)$$

$$\boxed{\omega_i = \omega_B = 0} \quad (26)$$

(Not rotating when it hits the horizontal plane)

$$\boxed{\omega_f = \omega_c = -\frac{v_c}{R_2}} \quad (27)$$

rolling clockwise w/o slipping

$$\boxed{\mathcal{J}_{ang,tot} = -R_2 \int_{t_i}^{t_f} f_k\, dt = +R_2 \mathcal{J}_x = -I \frac{v_c}{R_2}} \quad (28)$$

So we have

$$\boxed{\begin{aligned} \mathcal{J}_x &= M(v_c - v_B) \\ R_2 \mathcal{J}_x &= -I \frac{v_c}{R_2} \end{aligned}} \quad (29)$$

or

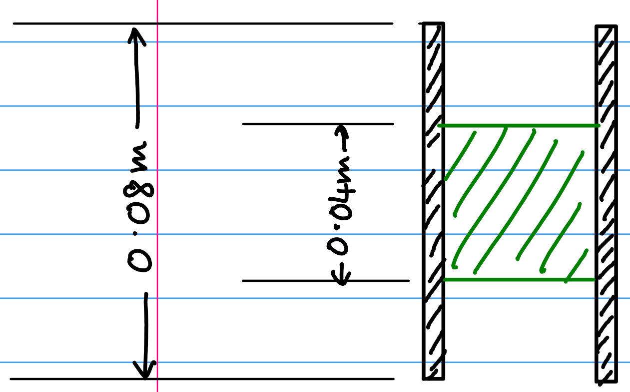
$$-I \frac{v_c}{R_2^2} = M(v_c - v_B)$$

$$\Rightarrow \quad v_c \left[1 + \frac{I}{MR_2^2}\right] = v_B$$

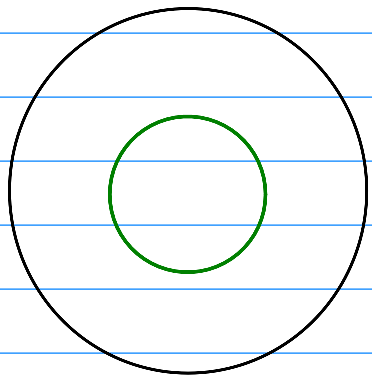
$$\boxed{v_c = \frac{v_B}{1 + \frac{I}{MR_2^2}} = \frac{7.27 \text{ m/s}}{1 + \frac{0.65}{0.9}}.} \quad (30)$$

$$\boxed{= 4.22 \text{ m/s}} \quad (31)$$

Example 2 : A yo-yo can be modelled as a solid central cylinder of mass $\boxed{0.2\,\text{kg}}$ (32) and radius $\boxed{0.02\text{ m}}$ (33), and two solid disks of mass $\boxed{0.1\,\text{kg}}$ (34) each and radius $\boxed{0.04\text{ m}}$ (35)



Side view


front view

The yoyo is allowed to fall 1m before its string tightens and then it starts unwinding the string (assumed to be massless)

1a: What is the moment of inertia of the yoyo?

Basically there are 3 solid cylinders, so

$$\boxed{\begin{aligned} I_{tot} &= \frac{1}{2}(0.2\,kg)(0.02m)^2 + 2 \times \frac{1}{2}(0.1kg)(0.04)^2 \\ &= 4\times10^{-5}\,kgm^2 + 16\times10^{-5}\,kgm^2 = 2\times10^{-4}\,kgm^2 \end{aligned}}$$ 36

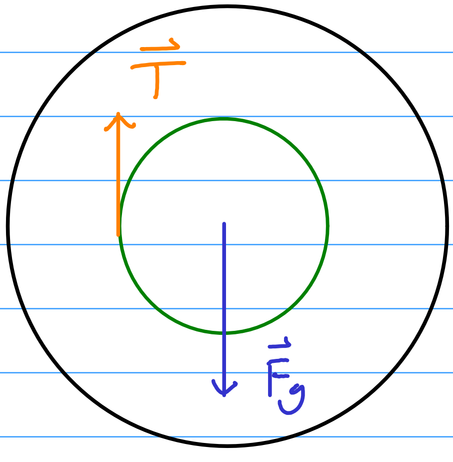
1b: What is the yoyo's speed just before the string tightens?

Since $\boxed{W_{nc}=0}$ 37 $\boxed{\Delta E_{mech}=0}$ 38

$\Rightarrow \quad Mg\,(1m) = \frac{1}{2}Mv_{cM}^2 \qquad$ (no rotation)

$\Rightarrow \quad \boxed{v_{cM} = \sqrt{2g\,(1m)} = 4.43\ m/s}$ 39

1c Find the CM acceleration once the string tightens



Only $\overline{T}$ produces torque around the CM. Apply the linear Newton II first

$$\boxed{T - Mg = -Ma_{CM}}$$ (40)

because $\vec{a} = -a_{CM}\hat{j}$

Now angular Newton II

$$\boxed{\tau_{tot} = -TR_1 = I\alpha}$$ (41)

Because the yoyo rolls w/o slipping on the string

$$\boxed{\alpha = -\frac{a_{CM}}{R_1}}$$ (42)

(clockwise acceleration a_{CM} positive)

$$\Rightarrow \boxed{-TR_1 = -I\frac{a_{CM}}{R_1} \quad \text{or} \quad T = \frac{I a_{CM}}{R_1^2}}$$ (43)

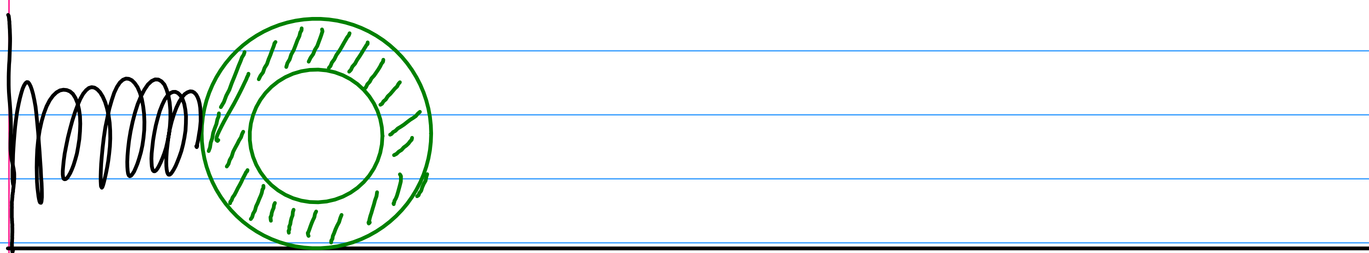
$$\Rightarrow \boxed{\frac{I a_{CM}}{R_1^2} - Mg = Ma_{CM}}$$ (44)

or $$a_{CM}\left[1 + \frac{I}{MR_1^2}\right] = g$$

$$\boxed{a_{CM} = \frac{g}{1 + \frac{I}{MR_1^2}}}$$ (45) $$= \frac{g}{1 + \frac{2\times10^{-4}\ \text{kgm}^2}{0.4\,\text{kg}\times 4\times10^{-4}}}$$

$$= \frac{g}{1 + 1.25} = \frac{9.8\ \text{m/s}^2}{2.25} = \boxed{4.35\ \text{m/s}^2}$$ (46)

Example 3 : A hollow sphere of Mass $\boxed{10kg}$, (47) inner radius $\boxed{0.2m}$ (48) and outer radius $\boxed{0.3m}$ (49) is held against a spring of force constant $\boxed{4000\ N/m}$ (50), compressed to $\boxed{x_i' = -0.2m}$ (51). It is released from the spring. Assume that it does not rotate as it leaves the spring. It then skids on a horizontal surface with friction until it starts rolling w/o slipping.



1a Find the moment of inertia of the hollow sphere.

Think of the hollow sphere as the superposition of a positive mass solid sphere of radius R_2 and Mass

$$\boxed{M_2 = \rho \frac{4}{3}\pi R_2^3}$$ (52)

and a negative mass solid sphere of radius R_1 and Mass

$$\boxed{M_1 = -\rho \frac{4}{3}\pi R_1^3}$$ (53)

$$\boxed{I_{tot} = \frac{2}{5} M_1 R_1^2 + \frac{2}{5} M_2 R_2^2}$$ (54)

$$= \boxed{\frac{2}{5}\rho \frac{4}{3}\pi \left[R_2^5 - R_1^5\right]}$$ (55)

$$\boxed{\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi (R_2^3 - R_1^3)}}$$ (56)

So $I_{tot} = \frac{2}{5} M \frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}$

(57)

$$= \frac{2}{5} \times 10\,kg \frac{\left[(0.3m)^5 - (0.2m)^5\right]}{(0.3m)^3 - (0.2m)^3} = \boxed{0.444\,kg\,m^2}$$

For future reference $\boxed{\frac{I}{MR_2^2} = 0.494}$ (58)

1b : what is the speed of the sphere when it leaves the spring?

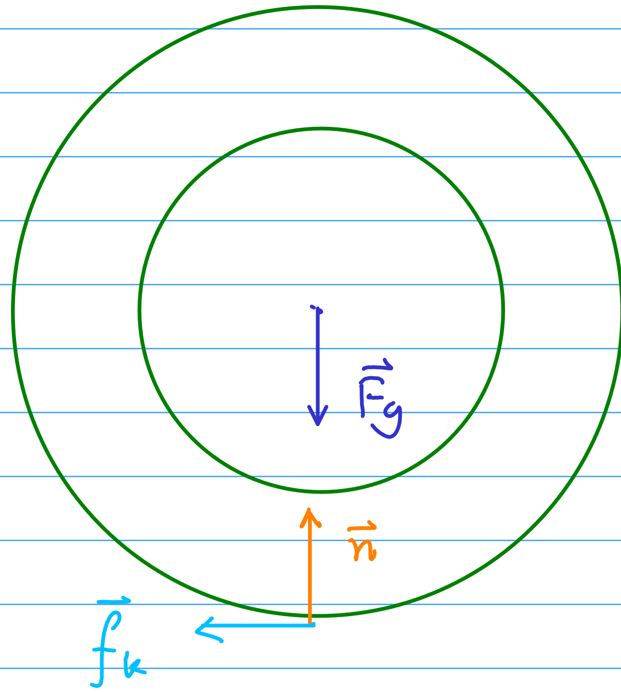
$\boxed{W_{nc} = 0}$ (59) $\Rightarrow$ $\boxed{E_{mech,A} = E_{mech,B}}$ (60)

$\Rightarrow \boxed{\frac{1}{2} k (x_i')^2 = \frac{1}{2} M v_B^2}$ (61) (No rotation)

$$\frac{1}{2} \times 4000 \frac{N}{m} (0.04\,m^2) = \frac{1}{2} \times 10\,kg\, v_B^2$$

$\boxed{v_B = 4\,m/s}$ (62)

1c: Use the linear and angular versions of the Impulse-Momentum theorem to find the CM speed of the sphere when it rolls w/o slipping.



$a_y = 0$

$\Rightarrow \boxed{n = Mg}$ 63

64 $\boxed{\vec{F}_{tot} = -f_k \hat{i}}$

Only f_k produces a torque about the CM.

Linear Impulse-momentum thm in the x-direction

$$\boxed{\mathcal{J}_x = \int_{t_i}^{t_f} (-f_k)\, dt = \Delta p_x = M(v_c - v_B)}$$ 65

Angular Impulse-momentum thm is 66

$$\boxed{\mathcal{J}_{ang} = \int_{t_i}^{t_f} \tau_{tot}\, dt = \Delta L = I(\omega_f - \omega_i)}$$

$\boxed{\omega_i = 0}$ 67

Now $\boxed{\tau_{tot} = -f_k R_2}$ (68) (clockwise torque)

When it finally rolls w/o slipping

$$\boxed{\omega_f = -\frac{v_c}{R_2}} \quad (69) \quad \text{(clockwise)}$$

So $\boxed{J_{ang} = +R_2 J_x = -I\frac{v_c}{R_2}}$ (70)

$$\Rightarrow \quad J_x = -\frac{I v_c}{R_2^2} = M(v_c - v_B)$$

(71)

or $\boxed{v_c = \frac{v_B}{1 + \frac{I}{MR_2^2}} = \frac{v_B}{1+0.494} = 2.677 \text{ m/s}}$