A complicated cop and speeder problem

A speeder is initially going 40 m/s when he passes a police car going the other way at 20 m/s. As soon as they pass each other the cop slams on the brakes and decelerates at 8 m/s². He comes to a stop and takes 1 sec to turn the police car around.
He puts on the flashers and accelerates after the speeder with $a = 4\,m/s^2$.
As soon as the speeder sees the flashers he starts decelerating at 2 m/s². Where and when do they meet, and what is each car's speed then?

Solution: There are 3 phases. In phase I the police car has a negative velocity but positive acceleration and this phase ends when the police car comes to rest. The speeder travels at constant velocity in this phase.

In phase II the police car turns around. Its x position does not change. The speeder has constant velocity in this phase

In phase III the police car has positive velocity and positive acceleration. The speeder has positive velocity and negative acceleration.

Assume that the point where they initially cross is $x=0$ and the time they cross is $t=0$

$C =$ Cop $\quad S =$ speeder

Phase I: $\boxed{v_{ic} = -20\,m/s}$ $\quad \boxed{a_c = 8\,m/s^2}$

$\boxed{v_{fc} = 0\,m/s}$

What is the time interval Δt_I?

$$\boxed{v_{fc} = v_{ic} + a_c \Delta t_I}$$

$$0 = -20\,\frac{m}{s} + 8\,\frac{m}{s^2}\,\Delta t_I \Rightarrow \boxed{\Delta t_I = 2.5\text{ sec}}$$

To find the cop's position use

$$\boxed{\Delta x_{,c} = v_{av,c}\Delta t_I} \quad \boxed{v_{av,c} = \frac{v_{ic}+v_{fc}}{2} = -10\,\frac{m}{s}}$$

$$\boxed{\Delta x_c = -10\,\frac{m}{s} \times 2.5\text{sec} = -25\,m.}$$

Since $\boxed{x_c(0)=0}$ we know $\boxed{x_c(2.5\text{sec}) = -25\,m}$

In the mean time the speeder travels at a constant velocity of $v_{is} = 40\,m/s$

$$\Rightarrow \boxed{\Delta x_s = 40\,\frac{m}{s} \times 2.5\text{ sec} = 100\,m}$$

$$\boxed{x_s(0)=0} \Rightarrow \boxed{x_s(2.5\text{sec}) = 100\,m}$$

Phase II : The cop's position is unchanged and his velocity is still 0. This phase takes 1 sec, so at the end of this phase $\boxed{t_2 = 3.5\ \text{sec}}$

$$\boxed{x_c(t_2) = x_c(3.5\ \text{sec}) = -25\ m}$$

$$\boxed{v_c(t_2) = v_c(3.5\ \text{sec}) = 0}$$

In this second the speeder has travelled another 40m. So

$$\boxed{x_s(t_2) = 140\ m}$$

$$\boxed{v_s(t_2) = 40\ m/s.}$$

Phase III The cop accelerates at $4 m/s^2$.

$$\Rightarrow \boxed{v_c(t_2 + \Delta t) = 4 \frac{m}{s^2} \otimes \Delta t}$$

Average speed between t_2 and $t_2 + \Delta t$ is

$$\boxed{v_{av,c}(t_2, t_2 + \Delta t) = 2 \frac{m}{s^2} \Delta t}$$

$$\Rightarrow \boxed{\Delta x_c = v_{av,c}(t_2; t_2 + \Delta t) \otimes \Delta t = 2 \frac{m}{s^2} (\Delta t)^2}$$

Since $x_c(t_2) = -25\ m$

$$\boxed{x_c(t_2 + \Delta t) = -25 + 2 (\Delta t)^2}$$

For the speeder $\boxed{a_s = -2\,\frac{m}{s^2}}$

$v_s(t_2) = 40\,\frac{m}{s} \Rightarrow \boxed{v_s(t_2+\Delta t) = 40 - 2\Delta t}$

average velocity of speeder between t_2 and $t_2+\Delta t$ is

$$\boxed{\begin{aligned} v_{av,s}(t_2;t_2+\Delta t) &= \frac{1}{2}(40 + 40 - 2\Delta t) \\ &= 40 - \Delta t \end{aligned}}$$

$$\boxed{\Delta x_s(t_2;t_2+\Delta t) = v_{av,s}(t_2;t_2+\Delta t) \otimes \Delta t = (40-\Delta t)\Delta t}$$

Since $x_s(t_2) = 140\,m$

$$\boxed{x_s(t_2+\Delta t) = 140 + 40\,\Delta t - (\Delta t)^2}$$

They meet when $\boxed{x_s(t_2+\Delta t) = x_c(t_1+\Delta t)}$

$\Rightarrow \quad -25 + 2(\Delta t)^2 = 140 + 40\,\Delta t - (\Delta t)^2$

or $\boxed{3(\Delta t)^2 - 40(\Delta t) - 165 = 0}$

Solve $\boxed{\Delta t = \frac{40 \pm \sqrt{(40)^2 - 4\,(3)\,(-165)}}{6}}$

$$= \frac{40 \pm \sqrt{1600 + 1980}}{6} = \frac{40 \pm \sqrt{3580}}{6}$$

$\sqrt{3580} \cong 59.8$

Clearly Δt must be positive so choose the + solution

$$\boxed{\Delta t = \frac{40 + 59.8}{6} = 16.6 \text{ sec}}$$

Where are they? Take $x_c(t_2 + \Delta t)$

$$\boxed{x_c(t_2 + \Delta t) = -25 + 2(16.6)^2 = 252 \text{ m}}$$

What are their velocities at this time?

$$\boxed{\begin{aligned} v_c(t_2 + \Delta t) &= a_{c,\mathrm{II}}\,\Delta t = 4\,\frac{\text{m}}{\text{s}^2}\,(16.6 \text{ sec}) \\ &= 66.4 \text{ m/s} \end{aligned}}$$

$$\boxed{\begin{aligned} v_s(t_2 + \Delta t) &= v_s(t_2) - 2\,\frac{\text{m}}{\text{s}^2}\,\Delta t \\ &= 40 - 2(16.6) = 7.8 \text{ m/s} \end{aligned}}$$