Practice Final Problems– Gravity

The key ideas are very few.

① Newton's law of Universal Gravitation

$$\boxed{\vec{F}_{12} = -\frac{G M_1 M_2}{r_{12}^2}\,\hat{r}_{12}} \quad (1)$$

$$\boxed{G = 6.67 \times 10^{-11}\ \frac{N m^2}{kg^2}} \quad (2)$$

$$\boxed{\vec{r}_{12} = \vec{r}_2 - \vec{r}_1} \quad (3)$$

$$\boxed{\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}} \quad (4)$$

② Gravity is conservative

$$\boxed{U_g(\vec{r}_{12}) = -\frac{G M_1 M_2}{r_{12}}} \quad (5)$$

So in any satellite motion, E_{mech} is conserved

③ Gravity is a central force, so if the CM of the two bodies is chosen as the origin the torque is zero

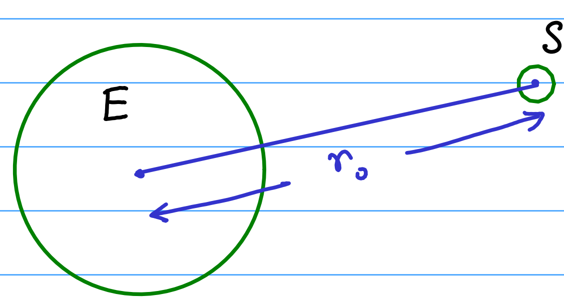
$$\Rightarrow \boxed{\vec{L} \text{ is conserved.}} \quad (6)$$

Example 1: A satellite of mass 400 kg (7) is initially in a low orbit with orbital period 2 hours. (8) A malfunction causes the satellite to explode into two masses, one of 100 kg (9) and the other of 300 kg. (10) It is observed that immediately after the collision the 100 kg mass is momentarily at rest, and subsequently falls (11) straight down to Earth.

$$GM_E = 4\times10^{14}\ Nm^2/kg$$

1a: Find the radius r_0 of the satellite's initial orbit and the speed v_0 before the explosion.

The only force acting on the satellite is the gravitational attraction of the Earth



(12)

$$F_s = \frac{G M_E M_s}{r_0^2} = \frac{M_s v_0^2}{r_0} \quad \text{(circular orbit)}$$

$$v_0 = \sqrt{\frac{GM_E}{r_0}} \quad (13) \quad \Rightarrow \quad T_0 = \frac{2\pi r_0}{v_0} = \frac{2\pi}{\sqrt{GM_E}}\, r_0^{3/2} \quad (14)$$

$$\boxed{r_0 = \left\{\frac{T_0^2\, GM_E}{4\pi^2}\right\}^{\frac{1}{3}}} \quad (15)$$

$$\boxed{T_0 = 2\text{hours} = 7200\,\text{sec} = 7.2\times10^3\,\text{sec}} \quad (16)$$

$$r_0 = \left\{\frac{7.2^2 \times 10^6 \times 4\times10^{14}}{4\pi^2}\right\}^{1/3} = \left\{0.5252\times10^{21}\right\}^{1/3}$$

$$\boxed{r_0 = 0.807\times10^7\,m = 8.07\times10^6\,m} \quad (17)$$

Note that $\boxed{R_E = 6.4\times10^6\,m}$ (18) so this is a really low orbit.

$$\boxed{v_0 = \frac{2\pi r_0}{T_0} = 7.04\times10^3\,m/s} \quad (19)$$

<u>1b</u>: Find the speed v_1 of the 300 kg fragment immediately after the explosion.

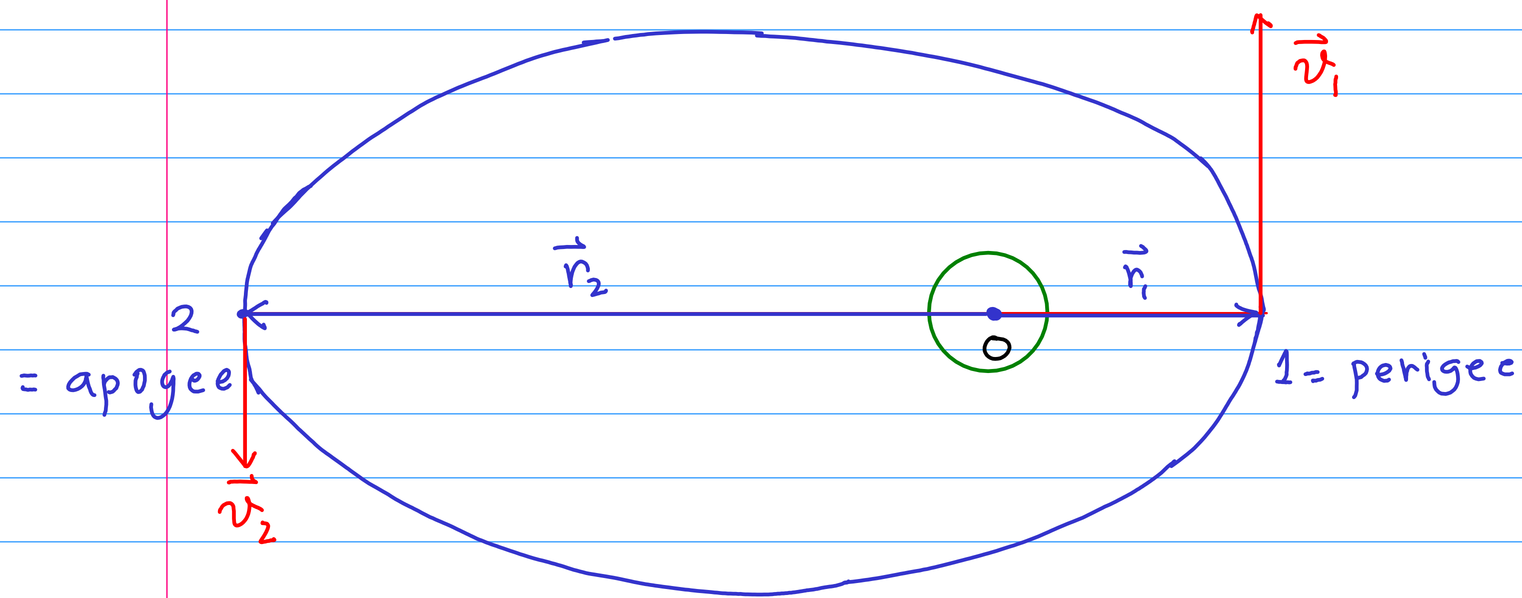
The explosion is an inelastic collision run backwards. So linear momentum is conserved. The 100kg fragment is stationary immediately after the explosion, so

$$\boxed{400\,kg \times 7.04\times10^3\,m/s = 300\,kg \cdot v_1} \quad (20)$$

$$\Rightarrow \boxed{v_1 = \frac{4}{3}\times 7.04\times10^3\,m/s = 9.39\times10^3\,m/s} \quad (21)$$

1c Use the conservation of Emech and $\vec{L}$ to find the distance of the 200 kg fragment at apogee.

Just after the explosion the satellite's velocity is $\perp$ to the position vector measured from the center of the Earth. So this point must be perigee.



Since angular momentum is conserved

$$\boxed{r_1 v_1 = r_2 v_2} \quad (22) \quad \Rightarrow \quad \boxed{\frac{1}{r_2} = \frac{v_2}{r_1 v_1}} \quad (23)$$

$r_1 = r_0 \qquad v_1 = \frac{4}{3} v_0 \qquad$ So

$$\boxed{\begin{aligned} r_1 v_1 &= 8.07 \times 10^6 \, m \times 9.39 \times 10^3 \, m/s \quad (24) \\ &= 7.58 \times 10^{10} \, m^2/s \end{aligned}}$$

Since E_{mech} is conserved

$$\boxed{\frac{1}{2} M v_1^2 - \frac{G M_E M}{r_1} = \frac{1}{2} M v_2^2 - \frac{G M_E M}{r_2}}$$ (25)

divide by $\frac{1}{2} M_1$ since it appears everywhere.

$$\boxed{v_1^2 - \frac{2 G M_E}{r_1} = v_2^2 - \frac{2 G M_E}{r_2}}$$ (26)

We can compute the LHS explicitly

$$(9.39 \times 10^3)^2 - \frac{2 \times 4 \times 10^{14}}{8.07 \times 10^6}$$

$$= 8.82 \times 10^7 \ \frac{J}{kg} - 9.91 \times 10^7 \ J/kg$$

$$\boxed{= -1.09 \times 10^7 \ J/kg}$$ (27)

Now plug in for $\frac{1}{r_2} = \frac{v_2}{v_1 r_1} = \frac{v_2}{7.58 \times 10^{10}}$

To make the numbers a bit more managable let us define

$$\boxed{v_2 = 10^3 \ \tilde{v}_2}$$ (28)

So $-1.09 \times 10^7 = 10^6 \ \tilde{v}_2^2 - \frac{2 \times 4 \times 10^{14} \times 10^3 \ \tilde{v}_2}{7.58 \times 10^{10}}$

or $-10.9 = \tilde{v}_2^2 - 10.55\,\tilde{v}_2$

$$\boxed{\tilde{v}_2^2 - 10.55\,\tilde{v}_2 + 10.9}$$ (29)

$$\boxed{\tilde{v}_2 = \frac{10.55 \pm \sqrt{10.55^2 - 43.6}}{2} = \frac{10.55 \pm 8.23}{2}}$$ (30)

$\boxed{\tilde{v}_2 = 9.39 \text{ m/s}}$ (31) or $\boxed{\tilde{v}_2 = 1.16 \text{ m/s}}$ (32)

The 1st answer is $v_1 = 9.39 \times 10^3$ m/s, so the correct answer for the speed at apogee is

$$\boxed{v_2 = 1.16 \times 10^3 \text{ m/s}}$$ (33)

Now we can figure out $r_2 = \dfrac{r_1 v_1}{v_2}$

$$\boxed{r_2 = \frac{7.58 \times 10^{10}}{1.16 \times 10^3} = 6.53 \times 10^7 \text{ m}}$$ (34)

Example 2: A satellite initially in low Earth orbit with $\boxed{T_a = 8000\ sec}$. 35 It is desired to put it into a higher orbit with $\boxed{T_b = 20000\ sec}$. 36

This is done in two stages. In the 1st stage the satellite is given a boost $\boxed{\Delta v_a}$ 37 so that it goes into an 38 elliptical orbit with the perigee being $\boxed{r_a}$ (radius of circular orbit with period T_a) and the apogee being at $\boxed{r_b}$ (radius of circular orbit with period T_b). 39 40

At apogee it is given another boost $\boxed{\Delta v_b}$ so that it goes into circular orbit at r_b.

29: First find r_a and r_b

$$\boxed{r_a = \left\{ \frac{T_a^2 G M_E}{4\pi^2} \right\}^{\frac{1}{3}} = \left\{ \frac{8^2 \times 10^6 \times 4 \times 10^{14}}{4\pi^2} \right\}^{\frac{1}{3}} = 8.656 \times 10^6\ m}$$ 41

$$\boxed{r_b = \left\{ \frac{T_b^2 G M_E}{4\pi^2} \right\}^{1/3} = 18.42 \times 10^6\ m}$$ 42

and the velocities of the corresponding circular orbits 43 44

$$\boxed{v_a = \frac{2\pi r_a}{T_a} = 6.8 \times 10^3\ \frac{m}{s}} \qquad \boxed{v_b = \frac{2\pi r_b}{T_b} = 5.79 \times 10^3\ \frac{m}{s}}$$

2b: Find the neccessary boosts Δv_a and Δv_b

Now let $\boxed{v_1 = v_a + \Delta v_a}$ (45). Since this is the perigee of the elliptical orbit

$$\boxed{r_1 = r_a} \quad (46)$$

During the elliptical orbit E_{mech} and $\vec{L}$ are conserved. Define

$$\boxed{\tilde{L} = r_1 v_1 = r_2 v_2} \quad (47)$$

$$\Rightarrow \quad (48) \ \boxed{v_1 = \frac{\tilde{L}}{r_1}} \quad \text{and} \quad \boxed{v_2 = \frac{\tilde{L}}{r_2}} \quad (49)$$

$$\boxed{\frac{2E_{mech}}{M} = v_1^2 - \frac{2GM_E}{r_1} = v_2^2 - \frac{2GM_E}{r_2}} \quad (50)$$

Define $\boxed{\tilde{E} = \frac{2E_{mech}}{M}}$ (51)

$$\tilde{E} = v_1^2 - \frac{2GM_E}{r_1} \quad \text{or} \quad \boxed{\tilde{E} = \frac{\tilde{L}^2}{r_1^2} - \frac{2GM_E}{r_1}} \quad (52)$$

So $\boxed{\tilde{E} r_1^2 + 2GM_E r_1 - \tilde{L}^2 = 0}$ (53)

There will be two solutions corresponding to r_1 and r_2, provided $\tilde{E} < 0$

$$\boxed{\tilde{E} = -|\tilde{E}|} \quad (54)$$

$$\boxed{|\tilde{E}|\, r_1^2 - 2GM_E r_1 + \tilde{L}^2 = 0} \quad (55)$$

$$\boxed{r_{\pm} = \frac{GM_E \pm \sqrt{(GM_E)^2 - |\tilde{E}|\,\tilde{L}^2}}{|\tilde{E}|}} \quad (56)$$

We want $\boxed{r_+ = r_2 = r_b}$ (57) (apogee)

$\boxed{r_- = r_1 = r_a}$ (58) (perigee)

So $$\boxed{r_a + r_b = \frac{2GM_E}{|\tilde{E}|} = (8.656 + 18.42)\times 10^6 \,m} \quad (59)$$

$$\Rightarrow |\tilde{E}| = \frac{2 \times 4 \times 10^{14}}{27.076 \times 10^6} = 2.955 \times 10^7 \, J/kg$$

$$\boxed{\tilde{E} = -2.955 \times 10^7 \, J/kg} \quad (60)$$

Now we know $r_1 = r_a$ and

$$\boxed{\tilde{E} = v_1^2 - \frac{2GM_E}{r_a}.} \quad (61)$$

$$v_1^2 = -2.955 \times 10^7 + \frac{2 \times 4 \times 10^{14}}{8.656 \times 10^6} = 6.287 \times 10^7 \left(\frac{m}{s}\right)^2$$

$$\Rightarrow \boxed{v_1 = 7.93 \times 10^3 \, m/s} \quad (62)$$

Similarly, from $r_2 = r_b = 18.42 \times 10^6 \, m$ we get

$$v_2^2 = \tilde{E}_2 + \frac{2GM_E}{r_b} = -2.955\times10^7 + \frac{2\times4\times10^{14}}{18.42\times10^6}$$

$$= 1.388\times10^7 \ (m/s)^2$$

$$\boxed{v_2 = 3.72 \times10^3 \ m/s}$$ 63

We know from the solution of 2a that

$v_a = 6.8\times10^3 \ m/s$ and now we have found that the perigee velocity for the elliptical orbit is

$$v_1 = 7.93\times10^3 m/s.$$

So $\boxed{\Delta v_a = v_1 - v_a = 1.13\times10^3 m/s}$ 64

Similarly $v_2 = 3.72\times10^3 m/s$ but the velocity needed for a circular orbit at r_b is

$$v_b = 5.79\times10^3 m/s$$

$\Rightarrow$ $\boxed{\Delta v_b = v_b - v_2 = 2.07\times10^3 m/s}$ 65

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Example 3: A satellite of mass $\boxed{500\,kg}$ is in a circular geostationary Earth orbit. It suffers a head-on collision with a meteor of mass $\boxed{100\,kg}$ 67 moving at a speed of $\boxed{4\times10^{3}\,m/s}$ 68 opposite to the direction of motion of the satellite. The collision is totally inelastic

3a: Find the radius r_0 and velocity v_0 of the geostationary orbit

$$\boxed{T_0 = 24\,\text{hours} = 86400\,\text{sec}} \quad 69$$

70

$$\boxed{r_0 = \left\{\frac{T_0^2 G M_E}{4\pi^2}\right\}^{1/3} = \left\{\frac{8.64^2\times10^8 \times 4\times10^{14}}{4\pi^2}\right\}^{1/3}}$$

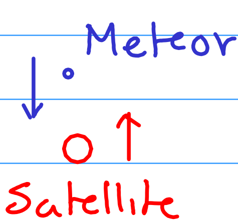
$$\boxed{r_0 = 4.23\times10^{7}\,m} \quad 71$$

72

$$\Rightarrow \boxed{v_0 = \frac{2\pi r_0}{T_0} = 3.075\times10^{3}\,m/s}$$

3b: Find the speed of satellite + meteor immediately after the collision.

Everything occurs in one dimension



$$\boxed{500\,kg \times 3.075\times10^3\,m/s - 100\,kg \times 4\times10^3\,m/s = (500\,kg + 100\,kg)\,v_f}$$ 73

$$\boxed{v_f = 1.9\times10^3\,m/s}$$ 74

3c: Find the perigee of the satellite after the collision.

Just after the collision the velocity is $\perp$ to $\vec{r}$, so it must be at apogee.

75 $\boxed{r_0 = r_2}$ $\boxed{v_f = v_2}$ 76

$$\boxed{\tilde{L} = \frac{L}{M} = r_2 v_2 = 4.23\times10^7\,m \times 1.9\times10^3\,m/s = 8.04\times10^{10}\,m^2/s}$$ 77

and
$$\boxed{\begin{aligned}\tilde{E} = \frac{2E_{mech}}{M_{tot}} &= v_2^2 - \frac{2GM_E}{r_2} \\ &= 1.9^2\times10^6 - \frac{2\times4\times10^{14}}{4.23\times10^7} \\ &= -1.53\times10^7\,J/kg\end{aligned}}$$ 78

By the conservation of E_{mech}, at perigee

$$\boxed{v_1^2 - \frac{2GM_E}{r_1} = \tilde{E}}$$ 79

$$\boxed{v_1 = \frac{\tilde{L}}{r_1}}$$ 80 $\Rightarrow$ $$\boxed{\frac{\tilde{L}^2}{r_1^2} - \frac{2GM_E}{r_1} = \tilde{E}}$$ 81

or $$-\tilde{E}\, r_1^2 - 2GM_E\, r_1 + \tilde{L}^2 = 0$$

$$\boxed{1.53\times10^7\, r_1^2 - 8\times10^{14}\, r_1 + 6.46\times10^{21} = 0} \quad (82)$$

let $\boxed{r_1 = 10^7\,\tilde{r}_1}$ (83)

$$\Rightarrow \quad \boxed{\tilde{r}_1^2 - 8\tilde{r}_1 + 6.46 = 0} \quad (84)$$

$$\boxed{\tilde{r}_1 = \frac{8 \pm \sqrt{64 - 4\times1.53\times6.46}}{2\times1.53} = \frac{8 \pm 4.95}{3.06}} \quad (85)$$

$\boxed{\tilde{r}_1 = 4.23}$ (86) or $\boxed{\tilde{r}_1 = 1}$ (87)

The 1st solution is actually r_2, so the correct answer for perigee is

$$\boxed{r_1 = 10^7\, m} \quad (88)$$

which implies $\boxed{v_1 = \frac{\tilde{L}}{r_1} = 8.04\times10^3\, m/s}$ (89)