The Harmonic Oscillator

Despite its simplicity the Harmonic Oscillator is one of the most important problems in QM. It serves as a springboard for more complicated problems.

We begin in 1D.

$$\boxed{\mathcal{H} = \frac{\mathbb{P}^2}{2M} + \frac{1}{2} M \omega_0^2 \mathbb{X}^2} \quad (1)$$

The standard undergraduate way to solve is to write it as a differential eq^n in real space.

We will first do it algebraically, and then connect to the wavefunctions.

Let us define an intrinsic length using the parameters available, $M, \omega_0, \hbar$

$$\frac{\hbar^2}{M \ell^2} = \hbar \omega_0 \Rightarrow \boxed{\ell = \sqrt{\frac{\hbar}{M\omega_0}}} \quad (2)$$

ℓ has dimensions of length, so the natural scale of momentum is $\hbar/\ell$

$$\Rightarrow \boxed{\mathcal{H} = \frac{1}{2M} \frac{\hbar^2}{\ell^2} \left(\frac{\ell \mathbb{P}}{\hbar} \right)^2 + \frac{1}{2} M \omega_0^2 \ell^2 \left(\frac{\mathbb{X}}{\ell} \right)^2} \quad (3)$$

$$\boxed{\mathcal{H} = \frac{\hbar\omega_0}{2}\left[\left(\frac{x}{\ell}\right)^2 + \left(\frac{\ell \mathbb{P}}{\hbar}\right)^2\right]} \quad (4)$$

There is a nice symmetry between x/ℓ and $\ell\mathbb{P}/\hbar$.

To take advantage of this we define the ladder operators a, a^+

$$\boxed{a = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell} + i\frac{\mathbb{P}\ell}{\hbar}\right) \qquad a^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell} - i\frac{\mathbb{P}\ell}{\hbar}\right)} \quad (5)$$

Clearly $\boxed{[a, a^+] = 1}$ (6)

a is also called a destruction operator, while a^+ is the creation operator.

Now $$\boxed{\frac{x}{\ell} = \frac{1}{\sqrt{2}}(a + a^+) \qquad \frac{\ell\mathbb{P}}{\hbar} = \frac{-i}{\sqrt{2}}(a - a^+)} \quad (7)$$

$$\mathcal{H} = \frac{\hbar\omega_0}{2}\left[\frac{1}{2}(a + a^+)(a + a^+) - \frac{1}{2}(a - a^+)(a - a^+)\right]$$

$$\boxed{\mathcal{H} = \frac{\hbar\omega_0}{2}\left[a^+a + aa^+\right]} \quad (8)$$

Use the commutator $[a, a^+] = 1$

$$\boxed{\mathcal{H} = \hbar\omega_0\left(a^+a + 1/2\right)} \quad (9)$$

Now let's investigate the relationship

between $\mathcal{H}$ and a, $a^\dagger$.

$a^\dagger a$ is a hermitian operator, so it will have real eigenvalues. Let's assume that we know the states

$$\boxed{a^\dagger a |\nu\rangle = \nu |\nu\rangle} \quad (10)$$

Now consider the state $\boxed{a|\nu\rangle}$ (11). Is this an eigenstate of $a^\dagger a$?

$$a^\dagger a \,(a|\nu\rangle) = (a^\dagger a \; a)|\nu\rangle$$

$$= a \; a^\dagger a |\nu\rangle + [a^\dagger a, a]|\nu\rangle \; = \; \nu\, a|\nu\rangle + [a^\dagger a, a]|\nu\rangle$$

$$[a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a]\, a = -a$$

$$\Rightarrow \quad \boxed{a^\dagger a \,(a|\nu\rangle) = (\nu - 1)\, a|\nu\rangle} \quad (12)$$

So $\boxed{a|\nu\rangle = c|\nu-1\rangle}$ (13) where c is a complex number. To find c we assume that $|\nu-1\rangle$ is normalized

$$\boxed{|c|^2 \langle \nu-1|\nu-1\rangle = |c|^2 = \langle\nu| a^\dagger a |\nu\rangle = \nu} \quad (14)$$

Since the norm of a state has to be positive we know that $\nu \geq 0$

But there is a problem: Given any $|\nu\rangle$ we can always hit it with a and and get $|\nu-1\rangle$.

How can we stop ν from going to negative numbers?

The only way to stop it is when ν are nonnegative integers. Consider the state $\nu=0$

$$\boxed{\begin{aligned} \langle 0|a^{\dagger}a|0\rangle &= 0 \\ \Rightarrow\ a|0\rangle &= 0 \end{aligned}} \quad (15)$$

Since the operator a "kills" the state giving the number zero, there is no way to proceed to get $\nu<0$

So, $\boxed{\nu = n \geq 0}$ (16) non-negative integers.

Now the energy of these states is, simply

$$\boxed{\mathcal{H}|n\rangle = \hbar\omega_0(n+\tfrac{1}{2})\,|n\rangle \Rightarrow E_n = \hbar\omega_0(n+\tfrac{1}{2})} \quad (17)$$

The state $|0\rangle$ is clearly the lowest energy state, or the ground state.

We already know something about $|0\rangle$, namely $a|0\rangle=0$. Let's write this in x-space

$$\boxed{\psi_0(x) = \langle x|0\rangle} \quad (18)$$

$$\boxed{\langle x|\,a\,|0\rangle = \langle x|\,\frac{1}{\sqrt{2}}\left(\frac{x}{\ell} + \frac{i\ell}{\hbar}p\right)|0\rangle = 0} \quad (19)$$

or

$$\boxed{\left[\frac{x}{\ell} + \frac{i\ell}{\hbar}\left(-i\hbar\frac{d}{dx}\right)\right]\psi_0(x) = 0}$$

or

$$\frac{d\psi_0}{dx} = -\frac{x}{\ell^2}\psi_0(x) \Rightarrow \boxed{\psi_0(x) = C\,e^{-\frac{x^2}{2\ell^2}}} \quad (20)$$

Let's normalize

$$\int_{-\infty}^{\infty} dx\, |C|^2 e^{-\frac{x^2}{\ell^2}} = 1 = |C|^2 \ell\sqrt{\pi}$$

$$\Rightarrow \boxed{\psi_0(x) = \frac{1}{\sqrt{\ell\sqrt{\pi}}}\, e^{-\frac{x^2}{2\ell^2}}} \quad (21)$$

Now let's try to obtain the excited states as well. Consider the state

$$\boxed{a^+|n\rangle} \quad (22)$$

This is also an eigenstate of a^+a

$$a^+a\,a^+|n\rangle = a^+a^+a|n\rangle + [a^+a, a^+]|n\rangle$$

Now $[a^+a, a^+] = [a^+, a^+]a + a^+[a, a^+] = a^+$

$$\text{so} \quad \boxed{a^+a\,a^+|n\rangle = (n+1)\,a^+|n\rangle} \quad (23)$$

$$\Rightarrow \quad \boxed{a^+|n\rangle = C|n+1\rangle.} \quad (24)$$

C is a complex number

Normalize $$\boxed{\begin{aligned}|C|^2 &= \langle n|a a^\dagger|n\rangle \\ &= \langle n|a^\dagger a + 1|n\rangle = n+1\end{aligned}}$$ (25)

$\Rightarrow$ can choose $C = \sqrt{n+1}$

So $$\boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle} \quad (26)$$

Starting from $|0\rangle$ (27)

$$\boxed{a^\dagger|0\rangle = |1\rangle \qquad a^\dagger|1\rangle = \sqrt{2}\,|2\rangle \qquad a^\dagger|2\rangle = \sqrt{3}\,|3\rangle \; \cdots}$$

or $$\boxed{|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle} \quad (28)$$

We can find the wavef^n's of these states as well.

To make things convenient, we define the dimensionless variable

$$\boxed{\xi = x/\ell} \quad (29)$$

$$\boxed{a^\dagger = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)} \quad (30) \qquad \psi_0(\xi) = \frac{1}{\sqrt{\ell\sqrt{\pi}}}\, e^{-\xi^2/2}$$

$$\boxed{\langle x|n\rangle = \frac{1}{\ell\sqrt{\pi}}\,\frac{1}{\sqrt{2^n n!}}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}} \quad (31)$$

Now $$\boxed{\left(\xi - \frac{d}{d\xi}\right)e^{\xi^2/2} = e^{\xi^2/2}\left(-\frac{d}{d\xi}\right)} \quad (32)$$

$$\langle x | n \rangle = \frac{1}{\sqrt{\ell\sqrt{\pi}\, 2^n n!}} \left(\xi - \frac{\partial}{\partial \xi}\right)^n e^{\frac{\xi^2}{2}} e^{-\xi^2}$$

Take the $e^{\xi^2/2}$ through each $\xi - \frac{\partial}{\partial \xi}$, converting it to $-\frac{\partial}{\partial \xi}$

$$\boxed{\begin{aligned} \langle x | n \rangle &= \frac{1}{\sqrt{\ell\sqrt{\pi}\, 2^n n!}} \frac{(-1)^n}{2^{n/2}} e^{\xi^2/2} \frac{d^n}{d\xi^n} e^{-\xi^2} \\ &= \frac{e^{-\xi^2/2}}{\sqrt{\ell\sqrt{\pi}}} \frac{1}{\sqrt{2^n n!}} \left[(-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \right] \end{aligned}} \quad (33)$$

The expression in the square brackets is the n^{th} Hermite polynomial

$$\boxed{H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}} \quad (34)$$

$H_0 = 1 \qquad H_1 = 2\xi \qquad H_2 = 4\xi^2 - 2 \qquad \dots$

In any eigenstate $|n\rangle$ of $\mathcal{H}$ we can calculate the mean and uncertainty of x and p.

$$\boxed{\langle n | x | n \rangle = \frac{\ell}{\sqrt{2}} \langle n | a + a^\dagger | n \rangle = 0} \quad (35)$$

$$\boxed{\langle n | x^2 | n \rangle = \frac{\ell^2}{2} \langle n | a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2} | n \rangle} \quad (36)$$

$$= \frac{\ell^2}{2} \langle n| 2a^\dagger a + 1 |n\rangle = \boxed{\left(n+\tfrac{1}{2}\right)\ell^2} \quad (37)$$

$$\boxed{\langle n| p |n\rangle = \frac{-i\hbar}{\ell\sqrt{2}} \langle n| a - a^\dagger |n\rangle = 0} \quad (38)$$

$$\boxed{\begin{aligned}\langle n| p^2 |n\rangle &= -\frac{\hbar^2}{2\ell^2} \langle n| a^2 - a a^\dagger - a^\dagger a + a^{\dagger 2} |n\rangle \\ &= \left(n+\tfrac{1}{2}\right) \frac{\hbar^2}{2\ell^2}\end{aligned}} \quad (39)$$

The time-evolution of an arbitrary state can be found by first expanding it in the energy eigenbasis

$$\boxed{|\psi\rangle = \sum_n \psi_n |n\rangle \Rightarrow |\psi(t)\rangle = \sum_n \psi_n |n\rangle e^{-i\omega_0 (n+\frac{1}{2}) t}} \quad (40)$$

According to the correspondence principle high quantum numbers are "classical". Let's see how this works by constructing a wave packet made of high values of n

From (35) we see that in order to have a nonzero mean value of x we need a superposition.

There is a very convenient set of states called "coherent states" which accomplish this and also have other nice properties.

Consider, for an arbitrary complex number z, the state

$$\boxed{|z\rangle = c\, e^{z a^{+}} |0\rangle} \quad (41)$$

Definition of the Coherent state

Let's normalize $|z\rangle$

$$\langle z|z\rangle = 1 = |c|^2 \langle 0| e^{z^* a} e^{z a^{+}} |0\rangle$$

$$= |c|^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^{*m}}{m!} \frac{z^n}{n!} \langle 0| a^m a^{+n} |0\rangle$$

Clearly we need consider only $m=n$

Now $\langle 0| a^m a^{+m} |0\rangle$

$$= m! \langle 0| \frac{a^m}{\sqrt{m!}} \frac{a^{+m}}{\sqrt{m!}} |0\rangle = m! \langle m|m\rangle = m!$$

$$\Rightarrow \langle z|z\rangle = |c|^2 \sum_{0}^{\infty} \frac{|z|^2}{m!} = |c|^2 e^{|z|^2}$$

$\Rightarrow$ Can choose $C = e^{-\frac{|z|^2}{2}}$

$$\boxed{|z\rangle = e^{-\frac{|z|^2}{2} + z a^{+}} |0\rangle} \quad (42)$$

The first nice feature of this state is that it is an eigenstate of a

$$a|z\rangle = e^{-\frac{|z|^2}{2}} a e^{za^\dagger}|0\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger}\left(e^{-za^\dagger} a e^{za^\dagger}\right)|0\rangle \quad (43)$$

Consider

$$A(z) = e^{-za^\dagger} a e^{za^\dagger} \quad (44)$$

$$\frac{dA}{dz} = e^{-za^\dagger}[a, a^\dagger] e^{za^\dagger} = 1 \quad (45)$$

$$\Rightarrow \quad A(z) = A(0) + z = a + z \quad (46)$$

$$\Rightarrow \quad a|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger}(a+z)|0\rangle = z|z\rangle \quad (47)$$

because $a|0\rangle = 0$

$|z\rangle$ is an eigenket of a

Take the adjoint of $a|z\rangle = z|z\rangle$

$$\Rightarrow \quad \langle z|a^\dagger = z^*\langle z| \quad (48)$$

So

$$\langle z|x|z\rangle = \frac{\ell}{\sqrt{2}}\langle z|a^\dagger + a|z\rangle = \frac{\ell}{\sqrt{2}}(z+z^*) \quad (49)$$

and

$$\langle z|p|z\rangle = \frac{-i\hbar}{\ell\sqrt{2}}\langle z|a - a^\dagger|z\rangle = \frac{-i\hbar}{\ell\sqrt{2}}(z-z^*) \quad (50)$$

$$\langle z|\mathcal{H}|z\rangle = \hbar\omega_0\langle z|a^\dagger a + \tfrac{1}{2}|z\rangle = \hbar\omega_0(|z|^2 + 1/2) = \bar{E} \quad (51)$$

$$\langle z|(\mathcal{H} - \bar{E})^2|z\rangle = (\hbar\omega_0)^2\langle z|(a^\dagger a\, a^\dagger a + a^\dagger a + \tfrac{1}{4})|z\rangle$$

$$-(\hbar\omega_0)^2\left(|z|^2 + \tfrac{1}{2}\right)^2 = (\hbar\omega_0)^2|z|^2$$

So $$\boxed{\Delta E = \hbar\omega_0 |z| \simeq \hbar\omega_0\sqrt{\frac{E}{\hbar\omega_0}}}$$ (52)

$$\langle z|x^2|z\rangle = \frac{\ell^2}{2}\langle z|a^{+2} + \underbrace{a a^+}_{} + a^+ a + a^2|z\rangle \qquad a^+a+1$$

$$= \frac{\ell^2}{2}\left(z^{*2} + 2z^* z + 1 + z^2\right) = \frac{\ell^2}{2}\left[(z+z^*)^2 + 1\right]$$

$$\boxed{(\Delta x)^2 = \ell^2/2}$$ (53)

$$\langle z|p^2|z\rangle = -\frac{\hbar^2}{2\ell^2}\langle z|a^2 + a^{+2} - a a^+ - a^+ a|z\rangle$$

$$= \frac{\hbar^2}{2\ell^2}\left(2|z|^2 - z^2 - z^{*2}\right) + \frac{\hbar^2}{2\ell^2}$$

$$\Rightarrow \boxed{(\Delta p)^2 = \frac{\hbar^2}{2\ell^2}}$$ (54) So $$\boxed{\Delta p\,\Delta x = \frac{\hbar}{2}}$$ (55)

Coherent states are minimum uncertainty packets.

Now consider the time-dependence of $|z\rangle$

$$\boxed{|z\rangle = e^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}\frac{z^m}{m!}a^{+m}|0\rangle = e^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}|m\rangle}$$ (56)

$$\Rightarrow \boxed{|z(t)\rangle = e^{-\frac{|z|^2}{2}}\sum_0^{\infty}\frac{z^m}{\sqrt{m!}}e^{-i(m+\frac{1}{2})\omega_0 t}|m\rangle}$$ (57)

$$= e^{-\frac{i\omega_0 t}{2} - \frac{|z|^2}{2}}\sum_0^{\infty}\frac{\left(z e^{-i\omega_0 t}\right)^m}{\sqrt{m!}}|m\rangle$$

$$\boxed{|z(t)\rangle = e^{-\frac{i\omega_0 t}{2}}\,|z e^{-i\omega_0 t}\rangle}$$ (58)

In other words the time-dependence leaves the state still as a coherent state, but makes $z \to z e^{-i\omega_0 t}$

let

$z = |z| e^{i\varphi}$

$$\Rightarrow \quad \boxed{\bar{x}(t) = \frac{\ell}{\sqrt{2}} \left(z(t) + z^*(t) \right)} \quad (59)$$

$$\boxed{\begin{aligned} \bar{x}(t) &= \ell \sqrt{2} |z| \cos(\omega_0 t - \varphi) \\ \bar{p}(t) &= \frac{\hbar}{\ell} \sqrt{2} \, |z| \sin(\omega_0 t - \varphi) \end{aligned}} \quad (60)$$

So, $|z(t)\rangle$ displays the classical oscillations of both x and p.

The coherent states are the closest analogues of classical states.

One not so nice property of coherent states is that they are overcomplete. There are too many of them and states with different $|z\rangle$, $|z'\rangle$ are not orthogonal.

You will explore some other properties in a homework.