Angular Momentum and Spin

1. $\vec{\mathbb{L}}$ as the generators of rotations

2. Algebra of $\mathbb{L}_i$ and its reperesentations; Spin

$\vec{\mathbb{L}}$ as generators of rotations

Rotations in 3D form an example of a mathematical object known as a group.

A group G with elements $g_1, g_2 \cdots$ satisfies

a) If $g_1, g_2 \in G$ then there is an operation called group multiplication and $g_1 \cdot g_2 \in G$ as well.

Note that $g_1 \cdot g_2$ is generically not $g_2 \cdot g_1$

b) There is a special element called the identity "e" such that

$$\boxed{g \cdot e = e \cdot g = g} \quad ①$$

c) Every element has an inverse belonging to G such that

$$\boxed{g \cdot g^{-1} = g^{-1} \cdot g = e} \quad ②$$

The group of rotations is an example of a continuous group, or a Lie group, in which one can choose several real parameters such that there is an element of the group for every set of parameters.

For rotations in 3 spatial dimensions we can specify a rotation in several ways. One simple way is to specify the axis of rotation (a unit vector $\hat{n}$) and the angle of rotation ψ around that axis.

We can call the rotation operator $\boxed{\hat{\mathbb{R}}(\hat{n},\psi)}$ (3)

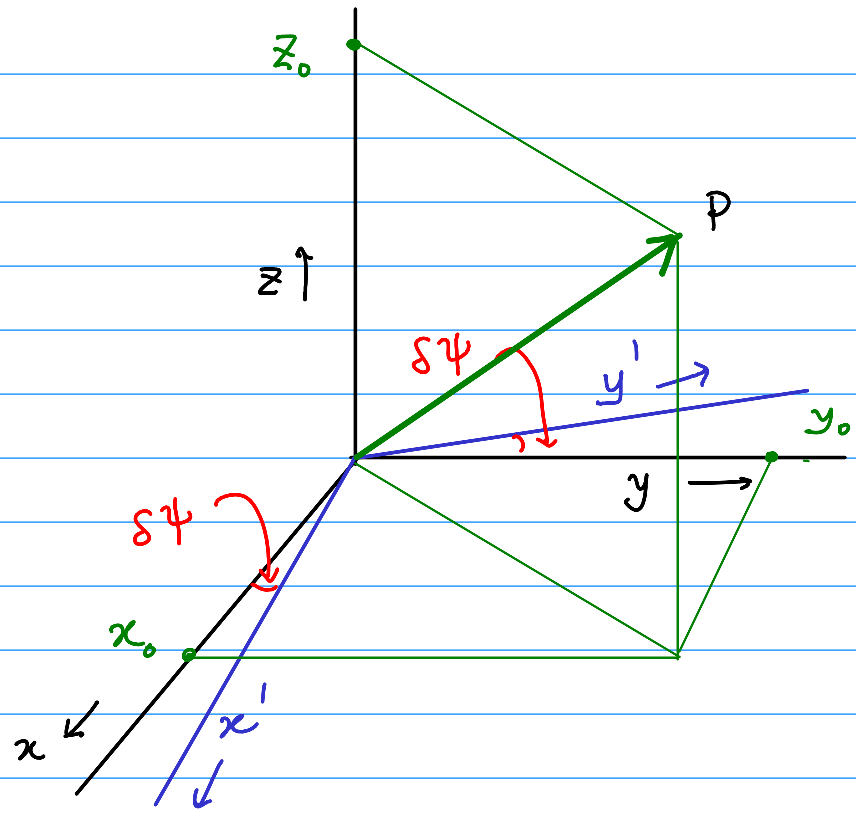
Clearly $\boxed{\hat{\mathbb{R}}(\hat{n},\underbrace{0}_{\psi=0}) = \mathbb{1}}$ (4) (Identity operator = "e" in the group)

It turns out to be extremely useful to consider infinitesimal transformations, i.e, where the angle of rotation ψ becomes $\delta\psi \ll 1$.

$$\boxed{\mathbb{R}(\hat{n},\delta\psi) = \mathbb{1} + \delta\psi\, \tilde{\mathbb{L}}(\hat{n}) + \cdots} \quad (5)$$

Let's consider a rotation around the z-axis $\theta = 0$ φ arbitrary

Suppose we have a point P which has coordinates (x_0, y_0, z_0) in one set of axes. We now rotate the axes and ask for the new coordinates of P (x_0', y_0', z_0')



Clearly
$$\begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} = \begin{pmatrix} x_0 + \delta\psi\, y_0 \\ y_0 - \delta\psi\, x_0 \\ z_0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta\psi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \qquad (6)$$

So we identify (7)

$$\tilde{\mathbb{L}}_z = \text{infinitesimal generator of rotations around } z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By very similar reasoning we can obtain

$$\boxed{\tilde{\mathbb{L}}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \tilde{\mathbb{L}}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad (8)$$

However, this is restricted to the action of rotations on a vector (the vector joining P to the origin).

There is another fruitful way to write $\mathbb{L}_z$ which does not suffer from this restriction

$$\boxed{\begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} = \begin{pmatrix} x_0 + \delta\psi\, y_0 \\ y_0 - \delta\psi\, x_0 \\ z_0 \end{pmatrix} = \left\{ \mathbb{1} + \delta\psi \left(y_0 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial y_0} \right) \right\} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}} \quad (9)$$

$$\boxed{\tilde{\mathbb{L}}_z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}} \quad (10)$$

This way of writing $\tilde{\mathbb{L}}_z$ works for arbitrary scalar functions $f(x,y,z)$.

In QM, we want unitary operators that act on states to represent spatial rotations. We will think of the functions that $\tilde{\mathbb{L}}_i$ and $\mathbb{R}$ act on as position-space wave functions.

As you can see, $\tilde{\mathbb{L}}_i$ is antihermitian and dimensionless. It is conventional to work with hermitian operators having dimensions of angular momentum by multiplying $\tilde{\mathbb{L}}_i$ by $i\hbar$

$$\mathbb{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad , \quad \mathbb{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\mathbb{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad (11)$$

Recalling that $\vec{\mathbb{p}} = -i\hbar\vec{\nabla}$ we see that these are the components of the quantum angular momentum operator.

$$\vec{\mathbb{L}} = \vec{\mathbb{x}} \times \vec{\mathbb{p}} \qquad (12)$$

The reason we call $\vec{\mathbb{L}}$ the generators of rotations is that we can construct any finite rotation by multiplying an infinite number of infinitesimal rotations

Unitary operator acting on Hilbert space states.

$$\mathbb{R}(\hat{n},\psi) = \lim_{N\to\infty}\left\{\mathbb{R}\left(\hat{n},\frac{\psi}{N}\right)\right\}^N = \lim_{N\to\infty}\left\{\mathbb{1} - \frac{i\psi}{\hbar N}\hat{n}\cdot\vec{\mathbb{L}}\right\}^N = e^{\frac{-i\psi\,\hat{n}\cdot\vec{\mathbb{L}}}{\hbar}} \qquad (13)$$

basically, exponentiating them. In this context, it is useful to recall the translation operator

$$\mathbb{T}_{\vec{a}}\,\psi(\vec{x}) = \psi(\bar{x}+\bar{a}) \qquad (14)$$

which can be written as a Taylor series

$$\psi(\bar{x}+\bar{a}) = \psi(\bar{x}) + \bar{a}\cdot\vec{\nabla}\psi(\bar{x}) + \frac{1}{2!}(\bar{a}\cdot\bar{\nabla})^2\psi(\bar{x}) + \cdots$$

$$= e^{\bar{a}\cdot\bar{\nabla}}\psi(\bar{x}) = e^{\frac{i\bar{a}\cdot\vec{\mathbb{p}}}{\hbar}}\psi(\bar{x}) \qquad (15)$$

Thus $\quad \mathbb{T}_{\bar{a}} = e^{\frac{i\bar{a}\cdot\vec{\mathbb{p}}}{\hbar}} \quad (16) \quad$ very similar to $\mathbb{R}$ in (13)

Algebra of $\vec{\mathbb{L}}$ and its representations

Given two group elements g_1 and g_2 the "group commutator" is defined as

$$\boxed{g_1 \cdot g_2 \cdot g_1^{-1} \cdot g_2^{-1} = g_c} \quad (17)$$

Suppose we take both g_1 and g_2 to be very close to $\mathbb{1}$

$$\boxed{g_1 = \mathbb{R}(\hat{n}_1, \delta\psi_1) \qquad g_2 = \mathbb{R}(\hat{n}_2, \delta\psi_2)} \quad (18)$$

Clearly g_c is also very close to $\mathbb{1}$

$$g_c = \left\{ \mathbb{1} - \frac{i}{\hbar} \delta\psi_1 \, \hat{n}_1 \cdot \vec{\mathbb{L}} + \cdots \right\} \left\{ \mathbb{1} - \frac{i}{\hbar} \delta\psi_2 \, \hat{n}_2 \cdot \vec{\mathbb{L}} + \cdots \right\} \otimes$$
$$\left\{ \mathbb{1} + \frac{i}{\hbar} \delta\psi_1 \, \hat{n}_1 \cdot \vec{\mathbb{L}} + \cdots \right\} \left\{ \mathbb{1} + \frac{i}{\hbar} \delta\psi_2 \, \hat{n}_2 \cdot \vec{\mathbb{L}} + \cdots \right\} \quad (19)$$

The leading order terms are

$$\boxed{g_c = \mathbb{1} - \frac{1}{\hbar^2} \delta\psi_1 \delta\psi_2 \left[\hat{n}_1 \cdot \vec{\mathbb{L}}, \hat{n}_2 \cdot \vec{\mathbb{L}} \right] + \cdots} \quad (20)$$

Thus the commutator of $\mathbb{L}_i$ and $\mathbb{L}_j$ must be proportional to other $\mathbb{L}$'s.

The commutator algebra of the generators of a Lie group are key to its representations.

It is easy to show that in our case

$$\boxed{[\mathbb{L}_i, \mathbb{L}_j] = i\hbar\, \epsilon_{ijk} \mathbb{L}_k} \quad (21)$$

where ϵ_{ijk} is the completely antisymmetric symbol.

Furthermore, defining

$$\boxed{\vec{\mathbb{L}}^2 = \mathbb{L}_x^2 + \mathbb{L}_y^2 + \mathbb{L}_z^2} \quad (22)$$

we can show $\boxed{[\mathbb{L}_i, \vec{\mathbb{L}}^2] = 0}$ (23) for all $i = x, y, z$

We can choose $\vec{\mathbb{L}}^2$ and one other $\mathbb{L}_i$ conventionally chosen as $\mathbb{L}_z$, as our two commuting operators. Let's consider states $|\lambda, \mu\rangle$ that are eigenstates of both $\vec{\mathbb{L}}^2$ & $\mathbb{L}_z$

$$\boxed{\vec{\mathbb{L}}^2 |\lambda, \mu\rangle = \hbar^2 \lambda(\lambda+1) |\lambda, \mu\rangle \quad ; \quad \mathbb{L}_z |\lambda, \mu\rangle = \hbar\mu |\lambda, \mu\rangle} \quad (24)$$

We assume $\boxed{\langle \lambda, \mu | \lambda, \mu\rangle = 1}$ (normalized) (25)

Now we define the ladder operators

$$\boxed{\mathbb{L}_\pm = \mathbb{L}_x \pm i\mathbb{L}_y.} \quad (26)$$

We can easily show $\boxed{[\mathbb{L}_z, \mathbb{L}_\pm] = \pm\hbar \mathbb{L}_\pm}$ (27)

and $\boxed{[\mathbb{L}_+, \mathbb{L}_-] = 2\hbar \mathbb{L}_z}$ (28)

(29)

Consider $\boxed{\mathbb{L}_+|\lambda,\mu\rangle}$. From (23) $[\vec{\mathbb{L}}^2, \mathbb{L}_\pm]=0$

so $\boxed{\vec{\mathbb{L}}^2\ \mathbb{L}_+|\lambda,\mu\rangle = \mathbb{L}_+\vec{\mathbb{L}}^2|\lambda,\mu\rangle = \hbar^2\lambda(\lambda+1)\ \mathbb{L}_+|\lambda,\mu\rangle}$ (30)

So $\mathbb{L}_+$ does not change λ. However, from (26)

$$\boxed{\mathbb{L}_z\ \mathbb{L}_+ = \mathbb{L}_+\mathbb{L}_z + \hbar\,\mathbb{L}_+} \quad (31)$$

$$\boxed{\begin{aligned}\mathbb{L}_z\ \mathbb{L}_+|\lambda,\mu\rangle &= \mathbb{L}_+\mathbb{L}_z|\lambda,\mu\rangle + \hbar\,\mathbb{L}_+|\lambda,\mu\rangle \\ &= \hbar(\mu+1)\,\mathbb{L}_+|\lambda,\mu\rangle \quad (32)\end{aligned}}$$

$\mathbb{L}_+$ increases the value of μ by 1. This implies

$$\boxed{\mathbb{L}_+|\lambda,\mu\rangle = \hbar\,C_+(\lambda,\mu)\,|\lambda,\mu+1\rangle} \quad (33)$$

where we assume $|\lambda,\mu+1\rangle$ is also normalized.

To find C_+ we take the norm of $\mathbb{L}_+|\lambda,\mu\rangle$

$$\boxed{\hbar^2|C_+(\lambda,\mu)|^2 = \langle\lambda,\mu|(\mathbb{L}_+)^\dagger\mathbb{L}_+|\lambda,\mu\rangle} \quad (34)$$

Now $\boxed{\mathbb{L}_+^\dagger = \mathbb{L}_-}$ (35)

Also
$$\boxed{\begin{aligned}\vec{\mathbb{L}}^2 &= \mathbb{L}_z^2 + \tfrac{1}{2}(\mathbb{L}_+\mathbb{L}_- + \mathbb{L}_-\mathbb{L}_+) \quad (36)\\ &= \mathbb{L}_z^2 + \hbar\,\mathbb{L}_z + \mathbb{L}_-\mathbb{L}_+ = \mathbb{L}_z^2 - \hbar\mathbb{L}_z + \mathbb{L}_+\mathbb{L}_-\end{aligned}}$$

$\Rightarrow$ $\boxed{\mathbb{L}_-\mathbb{L}_+ = \vec{\mathbb{L}}^2 - \mathbb{L}_z(\mathbb{L}_z+\hbar)}$ (37) $\boxed{\mathbb{L}_+\mathbb{L}_- = \vec{\mathbb{L}}^2 - \mathbb{L}_z(\mathbb{L}_z-\hbar)}$ (38)

Thus, from (34), (37)

$$\boxed{|C_+(\lambda,\mu)|^2 = \big(\lambda(\lambda+1) - \mu(\mu+1)\big)} \quad (39)$$

Since we can't have a negative norm we must demand that the maximum value of μ is λ

$$\boxed{(\mu)_{max} = \lambda} \quad (40)$$

for which $\boxed{\mathbb{L}_+ |\lambda,\lambda\rangle = 0}$ (41)

Now let's do the same for $\mathbb{L}_-$. It is easy to show that

$$\boxed{\mathbb{L}_- |\lambda,\mu\rangle = \hbar C_-(\lambda,\mu)\, |\lambda,\mu-1\rangle} \quad (42)$$

$$\boxed{\hbar^2 |C_-(\lambda,\mu)|^2 = \langle\lambda,\mu|\, \mathbb{L}_+ \mathbb{L}_- |\lambda,\mu\rangle} \quad (43)$$

$$\boxed{|C_-(\lambda,\mu)|^2 = \lambda(\lambda+1) - \mu(\mu-1)} \quad (44)$$

Again, since all norms are positive we must demand

$$\boxed{(\mu)_{min} = -\lambda} \quad (45)$$

and $\boxed{\mathbb{L}_- |\lambda,-\lambda\rangle = 0}$ (46)

Since we proceed in integer steps down from $\mu_{max} = +\lambda$

$$\boxed{\mu_{max} - \mu_{min} = 2\lambda = \text{integer.}} \quad (47)$$

$$\Rightarrow \boxed{\lambda = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots} \qquad (48)$$

Integer or half-odd integer.

The integer values correspond to the usual quantized orbital angular momenta, while the half-odd integers correspond to spin.

Each value of $\lambda \equiv j$ (henceforth integer or half-odd integer) and its values of $\mu \equiv m_j$ form a representation of the angular momentum algebra, and thus of the rotation group.

This basically means that upon a change of axes (rotation) a state $|j, m_j\rangle$ will change into a superposition of $|j, m_j'\rangle$

$$\boxed{\mathbb{R}(\hat{n}, \psi)\, |j, m_j\rangle = \sum_{m_j m_j'} D^{(j)}_{m_j m_j'}(\hat{n}, \psi)\, |j, m_j'\rangle} \qquad (49)$$

where the D's are some (in principle determined) complex numbers.

Different j's dont mix under rotations.

Let us explicitly consider the case $j = \frac{1}{2}$ and determine the rotation matrices.

$\mathbb{L}$ refers to orbital angular momentum. What we have is more general, capable of describing orbital and spin angular momentum.

Following convention, we denote these operators as $\vec{J}$

$$\boxed{[J_i, J_j] = i\hbar \epsilon_{ijk} J_k} \tag{50}$$

$$\boxed{\vec{J}^2 = J_x^2 + J_y^2 + J_z^2} \tag{51}$$

$$\boxed{\vec{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle \qquad J_z |j,m\rangle = \hbar m |j,m\rangle} \tag{52}$$

$$\boxed{j, m \text{ are integer or half-odd integer.}} \tag{53}$$

$$\boxed{J_\pm = J_x \pm i J_y} \tag{54}$$

$$\boxed{\begin{aligned} J_+ |j,m\rangle &= \hbar \sqrt{j(j+1) - m(m+1)}\; |j,m+1\rangle \\ J_- |j,m\rangle &= \hbar \sqrt{j(j+1) - m(m-1)}\; |j,m-1\rangle \end{aligned}} \tag{55}$$

Now let's focus on $j = \frac{1}{2}$. The only possible values of m are $\pm\frac{1}{2}$. The Hilbert space is two-dimensional. In this Hilbert space we can represent J_i and $\vec{J}^2$ as 2×2 matrices.

$$\boxed{\vec{J}^2 |\tfrac{1}{2}, m\rangle = \hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1)\, |\tfrac{1}{2}, m\rangle} \tag{56}$$

Let us label the Hilbert space as follows

$$\boxed{|\underset{j}{\tfrac{1}{2}}, \underset{m}{\tfrac{1}{2}}\rangle \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\underset{j}{\tfrac{1}{2}}, \underset{m}{-\tfrac{1}{2}}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \tag{57}$$

so
$$\boxed{J^2 \Rightarrow \frac{3}{4}\hbar^2 \begin{array}{cc|c} \frac{1}{2} & -\frac{1}{2} & \\ \hline 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{array} = \frac{3}{4}\hbar^2 \mathbb{1}} \quad (58)$$

Similarly
$$\boxed{J_z \Rightarrow \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar}{2}\sigma_z} \quad (59)$$

σ_z is a Pauli spin matrix.

Now consider
$$\boxed{J_x = \frac{J_+ + J_-}{2}} \quad (60)$$

$$J_x |\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{\hbar}{2} |\tfrac{1}{2},-\tfrac{1}{2}\rangle \qquad J_x |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{\hbar}{2} |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\Rightarrow \boxed{J_x \Rightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_x} \quad (61)$$

another Pauli matrix

$$\boxed{J_y = \frac{-i}{2}(J_+ - J_-)} \quad (62)$$

so

$$\boxed{J_y \Rightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_y} \quad (63)$$

The third Pauli matrix

These 3 matrices σ_i are extremely useful. It is worthwhile to familiarize yourself with their properties

$$\boxed{\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}} \quad (64) \qquad \boxed{\sigma_i^\dagger = \sigma_i} \quad (65)$$

(66)

$$\boxed{[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k} \qquad \boxed{\vec{J} = \frac{\hbar}{2}\vec{\sigma}} \quad (67)$$

They also satisfy an anticommutation relation. Define the anticommutator of 2 operators or matrices as

$$\boxed{\{A, B\} = AB + BA} \quad (68)$$

Then $$\boxed{\{\sigma_i, \sigma_j\} = 2\delta_{ij}\,\mathbb{1}} \quad (69)$$

This is the lowest dimensional example of a Clifford Algebra, which is useful in constructing the matrices which enter the Dirac eq^n.

Now, according to (13), in order to construct the rotation matrices $\mathbb{D}$, we need

$$\boxed{e^{-\frac{i}{\hbar}\psi\,\hat{n}\cdot\vec{J}} = e^{-i\frac{\psi}{2}\,\hat{n}\cdot\vec{\sigma}}} \quad (70)$$

Now $(\hat{n}\cdot\vec{\sigma})^2 = n_i\,\sigma_i\,n_j\,\sigma_j$

This is symmetric in i,j so

$$\boxed{(\hat{n}\cdot\vec{\sigma})^2 = \frac{1}{2}n_i n_j\,\{\sigma_i, \sigma_j\} = n_i n_j\,\delta_{ij}\,\mathbb{1} = \mathbb{1}} \quad (71)$$

because $\hat{n}$ is a unit vector

$$\boxed{e^{-i\frac{\psi}{2}\hat{n}\cdot\vec{\sigma}} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\frac{\psi}{2}\right)^k(\hat{n}\cdot\vec{\sigma})^k} \quad (72)$$

Divide into even and odd powers. (74)

$$\boxed{(\hat{n}\cdot\vec{\sigma})^{2k} = \mathbb{1}} \quad (73) \qquad \boxed{(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}}$$

$$\boxed{e^{-i\frac{\psi}{2}\hat{n}\cdot\vec{\sigma}} = \cos\left(\frac{\psi}{2}\right)\mathbb{1} - i\,\hat{n}\cdot\vec{\sigma}\,\sin\left(\frac{\psi}{2}\right)} \quad (75)$$

Denoting $\hat{n} = \sin\theta(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) + \cos\theta\,\hat{k}$

$$\boxed{\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix}} \quad (76)$$

$\Rightarrow$ the rotation matrix $\mathbb{D}^{(\frac{1}{2})}(\hat{n})$ is

$$\boxed{\mathbb{D}^{(1/2)}(\hat{n},\psi) = \begin{bmatrix} \cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\cos\theta & -i\sin\frac{\psi}{2}\sin\theta\, e^{-i\varphi} \\ -i\sin\frac{\psi}{2}\sin\theta\, e^{i\varphi} & \cos\frac{\psi}{2} + i\sin\frac{\psi}{2}\cos\theta \end{bmatrix}} \quad (77)$$

Let's consider a few examples of rotations.

First, let's rotate around the z axis by ψ

$\Rightarrow \hat{n} = \hat{z} \Rightarrow \theta = 0 \quad \varphi = \text{arbitrary}$

$$\boxed{\mathbb{D}^{(1/2)}(\hat{z},\psi) = \begin{bmatrix} \cos\frac{\psi}{2} - i\sin\frac{\psi}{2} & 0 \\ 0 & \cos\frac{\psi}{2} + i\sin\frac{\psi}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\psi}{2}} & 0 \\ 0 & e^{i\frac{\psi}{2}} \end{bmatrix}} \quad (78)$$

Here is an interesting fact: If $\psi = 2\pi$ we expect "nothing" to happen, which means the states should return to themselves.

However
$$\boxed{\mathbb{D}^{(1/2)}(\hat{z}, 2\pi) = -\mathbb{1}} \quad (79)$$

This is a bit disconcerting. However, recall that in QM we can multiply all states by the same phase without altering any measurable properties, so the states have returned to themselves.

Now consider a rotation of ψ around the x axis

$$\hat{n} = \hat{x} \quad \Rightarrow \quad \theta = \frac{\pi}{2} \quad \varphi = 0$$

$$\boxed{\mathbb{D}^{(1/2)}(\hat{x}, \psi) = \begin{bmatrix} \cos\frac{\psi}{2} & -i\sin\frac{\psi}{2} \\ -i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}} \quad (80)$$

Once again, if $\psi = 2\pi$ $\boxed{\mathbb{D}^{(1/2)}(\hat{x}, 2\pi) = -\mathbb{1}.}$ (81)

What if $\psi = \pi$? Naively, we expect that since the direction of z has reversed we should have $|\frac{1}{2}, \frac{1}{2}\rangle \rightarrow |\frac{1}{2}, -\frac{1}{2}\rangle$ and vice versa

$$\boxed{\mathbb{D}^{(1/2)}(\hat{x},\pi) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}} \quad (82)$$

So using (49), in the rotated frame (83)

$$\boxed{\begin{aligned} \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' &= \mathbb{D}^{(1/2)}_{11}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \mathbb{D}^{(1/2)}_{12}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -i\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \\ \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle' &= \mathbb{D}^{(1/2)}_{21}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \mathbb{D}^{(1/2)}_{22}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -i\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \end{aligned}}$$

Our naive intuition is correct up to a phase.

Now consider $j=1$. Now the Hilbert space is 3-dimensional. Let us label

$$\boxed{|1,1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad |1,0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |1,-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \quad (84)$$

Then it is easy to show

$$\boxed{\frac{J_x}{\hbar} \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \qquad \frac{\mathbb{J}_y}{\hbar} \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{bmatrix} \qquad \frac{\mathbb{J}_z}{\hbar} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \quad (85)$$

and of course $\boxed{\vec{\mathbb{J}}^{\,2} = 2\hbar^2 \mathbb{1}}$ (86)

Once again, we want $e^{-i\frac{\psi}{\hbar}\hat{n}\cdot\vec{\mathbb{J}}}$, which you will do in detail in the homework.

Another way to understand these states a bit better is to obtain wave functions.

Now, only integer j produce single-valued wave functions. The reason is that even though we intuitively think of spin as angular momentum, there is no θ, φ associated with spin. If we insist on finding a real-space wave f^n for spin, it will have various pathological properties, such as non-single-valuedness and divergences.

In the following, we will focus on $j=1$

In spherical polar coordinates r, θ, φ

$$\boxed{\mathbb{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \qquad \mathbb{L}_{\pm} = \hbar e^{\pm i\varphi}\left[\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi}\right]} \quad (87)$$

and
$$\boxed{\vec{\mathbb{L}}^2 = -\hbar^2 \left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right\}} \quad (88)$$

From $\boxed{\mathbb{L}_z |j,m\rangle = \hbar m |j,m\rangle}$ (89) we see that

$$\boxed{\psi_{j,m}(\theta,\varphi) \sim e^{im\varphi}\, \Phi_{j,m}(\theta)} \quad (90)$$

We seek the form of the $m=-j$ state obeying (46)

$$L_- \psi_{j,-j}(\theta,\varphi) = 0$$

$$\boxed{e^{-i\varphi}\left\{-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right\}\Phi_{j,-j}(\theta)\, e^{-ij\varphi} = 0} \quad (91)$$

or
$$-\frac{d\Phi_{j,-j}}{d\theta} + j\cot\theta\,\Phi_{j,-j} = 0$$

$$\boxed{\frac{d\Phi}{\Phi} = j\cot\theta\, d\theta = j\frac{d(\sin\theta)}{\sin\theta}} \quad (92)$$

$$\Rightarrow \quad \boxed{\Phi_{j,-j} = \text{constant}\,(\sin\theta)^j} \quad (93)$$

$$\Rightarrow \quad \boxed{\psi_{j,-j}(\theta,\varphi) = C_{j,-j}\,(\sin\theta)^j e^{-ij\varphi}} \quad (94)$$

This is true in general. For $j=1$

$$\boxed{\psi_{1,-1}(\theta,\varphi) = C_{1,-1}\sin\theta\, e^{-i\varphi}} \quad (95)$$

Normalizing over the sphere we find

$$|C_{1,-1}|^2 \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\, \sin^2\theta = 1 = \frac{8\pi}{3}|C_{1,-1}|^2$$

Choose $\boxed{C_{1,-1} \text{ real} = \sqrt{\frac{3}{8\pi}}}$ (96)

This is the Condon-Shortley phase convention

$$\boxed{\psi_{1,-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi}} \quad (97)$$

Now we can find $\psi_{1,0}$ and $\psi_{1,1}$ by successively applying (33), (39)

$$\boxed{\mathbb{L}_+ \psi_{j,m} = \hbar \sqrt{j(j+1) - m(m+1)}\; \psi_{j,m+1}} \quad (98)$$

$$\Rightarrow \boxed{e^{i\varphi}\left\{\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi}\right\} \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi} = \sqrt{2}\; \psi_{1,0}} \quad (99)$$

$$\boxed{\psi_{1,0} = \sqrt{\frac{3}{16\pi}} \left\{\cos\theta + \cos\theta\right\} = \sqrt{\frac{3}{4\pi}}\, \cos\theta} \quad (100)$$

$$\boxed{\mathbb{L}_+ \psi_{1,0} = \hbar\sqrt{2}\; \psi_{1,1}} \quad (101)$$

$$\Rightarrow \boxed{\psi_{1,1} = \frac{1}{\sqrt{2}}\, e^{i\varphi} \left\{\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right\} \sqrt{\frac{3}{4\pi}}\, \cos\theta} \quad (102)$$

$$\boxed{\psi_{1,1} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\varphi}} \quad (103)$$

These are exactly the spherical harmonics

$$\boxed{\begin{aligned} &\psi_{1,-1} = Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi} \qquad \psi_{1,0} = Y_{1,0} = \sqrt{\frac{3}{4\pi}}\, \cos\theta \\ &\psi_{1,1} = Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\varphi} \end{aligned}} \quad (104)$$

The $j=1$ representation is also called the vector representation. To understand this we combine $Y_{1,1}$ and $Y_{1,-1}$ to obtain

$$\boxed{\frac{\left(-Y_{1,1}+Y_{1,-1}\right)}{\sqrt{2}} = \sqrt{\frac{3}{4\pi}}\,\sin\theta\cos\varphi = \sqrt{\frac{3}{4\pi}}\,\frac{x}{r}} \quad (105)$$

$$\boxed{\frac{-Y_{1,1}-Y_{1,-1}}{i\sqrt{2}} = \sqrt{\frac{3}{4\pi}}\,\sin\theta\sin\varphi = \sqrt{\frac{3}{4\pi}}\,\frac{y}{r}} \quad (106)$$

$$\boxed{Y_{1,0} = \sqrt{\frac{3}{4\pi}}\,\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}} \quad (107)$$

The linear combinations above are the three Cartesian components of a 3D vector

Similarly the five components of $j=2$ $(m=-2,-1,0,1,2)$ correspond to the five independent elements of a traceless symmetric second rank tensor, an example being

$$\boxed{Q_{ij} = x_i x_j - \frac{1}{3}\vec{x}^2\delta_{ij}} \quad (108)$$