Addition of Angular Momenta

Let us start with a simple example. Say we have two 3D vectors $\vec{A}$ and $\vec{B}$. We know from the previous section that each transforms like $j=1$.

How many ways are there of multiplying them to get objects that transform properly under rotations?

$A_i B_j$ has 9 independent numbers. We are going to decompose these 9 into irreducible representations of rotations.

Firstly, we can take their dot product $\vec{A}\cdot\vec{B}$. This is a scalar, and is invariant under rotations and thus corresponds to $j=0$.

Secondly we can take the cross product

$$\boxed{\vec{C} = \vec{A}\times\vec{B} \qquad C_i = \epsilon_{ijk} A_j B_k} \quad (1)$$

This is a vector, and corresponds to $j=1$ and has 3 components. It is antisymmetric under $\vec{A} \leftrightarrow \vec{B}$

Finally we can take the symmetric product

$$\boxed{S_{ij} = A_i B_j + A_j B_i} \quad (2)$$

The trace of this is $2\vec{A}\cdot\vec{B}$ which we have already separated out. So the remaining object is a traceless, symmetric, second-rank tensor

$$\boxed{\tilde{S}_{ij} = A_i B_j + A_j B_i - \frac{2}{3}\delta_{ij}\,\vec{A}\cdot\vec{B}} \quad (3)$$

It has 5 independent components. Note that $9 = 1+3+5$, so we have decomposed $A_i B_j$ into a scalar $(j=0)$, a vector $(j=1)$ and a symmetric, traceless 2^{nd}-rank tensor $(j=2)$

Let's generalize this. Suppose we have two angular momenta $\vec{J}_1$ and $\vec{J}_2$ with states labelled by the $(2j_1+1)$ states $|j_1, m_1\rangle$ and the $(2j_2+1)$ states $|j_2, m_2\rangle$. $\vec{J} = \vec{J}_1 + \vec{J}_2$ clearly satisfies angular momentum commutators.

We "multiply" them, meaning we take the tensor product of the two Hilbert spaces to obtain the $(2j_1+1)\times(2j_2+1)$ states

$$\boxed{|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1, m_1; j_2, m_2\rangle} \quad (4)$$

We want to decompose them into multiplets that have definite $|j, m\rangle$. The number j now refers to the eigenvalue of the total angular momentum squared

$$\boxed{\vec{J}^{\,2}|j,m\rangle = (\vec{J}_1 + \vec{J}_2)^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle} \quad (5)$$

and m refers to total J_z

$$\boxed{J_z|j,m\rangle = (J_{1z}+J_{2z})|j,m\rangle = \hbar m|j,m\rangle} \quad (6)$$

The maximum value of m is clearly

$$\boxed{\max(m) = \max(m_1)+\max(m_2) = j_1+j_2} \quad (7)$$

So $\boxed{j_{max} = j_1+j_2}$ (8). It will turn out that $\boxed{j_{min} = |j_1-j_2|}$ (9). Each value of j between j_{min} and j_{max} separated by integers appears exactly once. The total number of states is

$$\boxed{N_{state} = \sum_{j=|j_1-j_2|}^{j_1+j_2}(2j+1) = \sum_{0}^{j_1+j_2}(2j+1) - \sum_{0}^{|j_1-j_2|-1}(2j+1)} \quad (10)$$

Using $\displaystyle\sum_{n=0}^{N} n = \frac{n(n+1)}{2}$ we get

$$\sum_{n=0}^{N}(2n+1) = (N+1)^2$$

and

$$\boxed{\begin{aligned} N_{state} &= (j_1+j_2+1)^2 - (j_1-j_2)^2 \quad (11) \\ &= (j_1+j_2)^2 + 2j_1+2j_2+1 - (j_1-j_2)^2 \\ &= 2j_1+2j_2+1+4j_1j_2 = (2j_1+1)(2j_2+1) \end{aligned}}$$

which is the number in the tensor product.

Thus

$$\boxed{|j,m;j_1,j_2\rangle = \sum_{m_1,m_2} C(j,m;j_1,m_1,j_2,m_2)\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle} \quad (12)$$

where the c's are the Clebsch-Gordan coefficients.

(13)

$$\boxed{C(j,m;j_1,m_1,j_2,m_2) = \langle j_1,m_1,j_2 m_2 | j,m;j_1,j_2\rangle}$$

Seen this way, it is clear that this is just a change of basis, from the one that diagonalizes, $\vec{J}_1^{\,2}$, $\vec{J}_2^{\,2}$, J_{1z} and J_{2z} to the one that diagonalizes

$$\vec{J}_1^{\,2},\ \vec{J}_2^{\,2},\ (\vec{J}_1+\vec{J}_2)^2 \text{ and } J_{1z}+J_{2z}$$

The simplest way to see how to obtain the CG coefficients is to illustrate with an example.

Say $j_1 = j_2 = 1$. We know that $\max(j) = 2$

So $|j,m;j_1,j_2\rangle = |2,2;1,1\rangle$ has to be

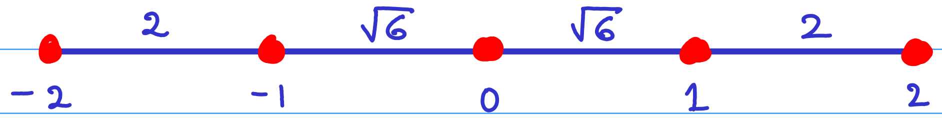
$|j_1,m_1\rangle \otimes |j_2,m_2\rangle = |1,1\rangle \otimes |1,1\rangle$ up to a phase.

In the Condon-Shortley phase convention this phase is 1. Thus

$$\boxed{|2,2;1,1\rangle = |1,1\rangle \otimes |1,1\rangle} \quad (14)$$

In the following, we will omit the j_1, j_2 everywhere and write

$$\boxed{\underset{j\ \ m}{|2,2\rangle} = \underset{m_1\ m_2}{|1,1\rangle}} \quad (15)$$



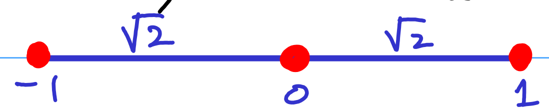
The "diagram" for $j=2$

Now lets apply $\boxed{J_- = J_{1-} + J_{2-}}$ (16) to this state

On the LHS

$$\boxed{\frac{1}{\hbar} J_- \underset{j\ \ m}{|2,2\rangle} = \underset{j(j+1)-m(m-1)}{\sqrt{2(2+1)-2(2-1)}}\ \underset{j,m-1}{|2,1\rangle} = 2|2,1\rangle} \quad (17)$$

On the RHS J_{1-} acts only on m_1, while J_{2-} acts only on m_2.



The $j=1$ "diagram"

$$\frac{1}{\hbar} J_{1-} \underset{m_1}{|1\rangle} = \underset{j_1(j_1+1)-m_1(m_1-1)}{\sqrt{1(1+1)-1(1-1)}}\ \underset{m_1-1}{|0\rangle} = \sqrt{2}\,|0\rangle$$

So
$$\boxed{\begin{aligned}\frac{1}{\hbar}\left(J_{1-}+J_{2-}\right)\underset{m_1\ m_2}{|1,1\rangle} &= \frac{1}{\hbar}J_{1-}|1,1\rangle + \frac{1}{\hbar}J_{2-}|1,1\rangle \\ &= \sqrt{2}\,|0,1\rangle + \sqrt{2}\,|1,0\rangle\end{aligned}} \quad (18)$$

Thus, equating the RHS and the LHS we get

$$2\underset{j\ \ m}{|2,1\rangle} = \sqrt{2}\left(\underset{m_1 m_2}{|0,1\rangle} + \underset{m_1\ m_2}{|1,0\rangle}\right)$$

$$\Rightarrow \quad \boxed{\underset{j\ \ m}{|2,1\rangle} = \frac{1}{\sqrt{2}}\left(\underset{m_1\ m_2}{|0,1\rangle} + \underset{m_1\ m_2}{|1,0\rangle}\right)} \quad (19)$$

Clearly, we can construct $|2,0\rangle$, $|2,-1\rangle$ and $|2,-2\rangle$ by successive application of J_-.

Before we do so, an important point must be noted. There are two tensor product states with $m_1+m_2=1$, which are $|1,0\rangle$ and $|0,1\rangle$.
The linear combination we constructed above belongs to the $j=2$ multiplet. Therefore, the othogonal linear combination must belong to the $j=1$ multiplet.

$$\Rightarrow \quad \boxed{\underset{j\ \ m}{|1,1\rangle} = \frac{1}{\sqrt{2}}\left(\underset{m_1\ m_2}{|1,0\rangle} - \underset{m_1\ m_2}{|0,1\rangle}\right)} \qquad (20)$$

This phase is also assigned in the Condon-Shortley convention.

Let's go back and obtain $|2,0\rangle$, $|2,-1\rangle$ and $|2,-2\rangle$

$$\boxed{\frac{J_-}{\hbar}\underset{j\ \ m}{|2,1\rangle} = \underset{j(j+1)-m(m-1)}{\sqrt{2(2+1)-1(1-1)}}\ \underset{j,\,m-1}{|2,0\rangle} = \sqrt{6}\,|2,0\rangle} \qquad (21)$$

On the RHS we get (22)

$$\boxed{\left(J_{1-}+J_{2-}\right)\frac{1}{\sqrt{2}}\left(\underset{m_1\ m_2}{|1,0\rangle}+\underset{m_1\ m_2}{|0,1\rangle}\right) = \frac{1}{\sqrt{2}}\Big\{J_{1-}|1,0\rangle + J_{1-}|0,1\rangle + J_{2-}|1,0\rangle + J_{2-}|0,1\rangle\Big\}}$$

$$= \frac{1}{\sqrt{2}}\left\{ \sqrt{2}\,|0,0\rangle + \sqrt{2}\,|-1,1\rangle + \sqrt{2}\,|1,-1\rangle + \sqrt{2}\,|0,0\rangle \right\}$$

$$\Rightarrow \quad \boxed{\underset{j\ \ m}{|2,0\rangle} = \frac{1}{\sqrt{6}}\left\{ \underset{m_1\ m_2}{|1,-1\rangle} + 2\underset{m_1\ m_2}{|0,0\rangle} + \underset{m_1\ m_2}{|-1,1\rangle} \right\}} \quad (23)$$

Apply J_- again. On the LHS

$$J_-|2,0\rangle = \sqrt{6}\,|2,-1\rangle$$

On the RHS

$$(J_{1-} + J_{2-})\frac{1}{\sqrt{6}}\left\{ |1,-1\rangle + 2\,|0,0\rangle + |-1,1\rangle \right\}$$

$$= \frac{1}{\sqrt{6}}\left\{ \sqrt{2}\,|0,-1\rangle + 2\sqrt{2}\,|-1,0\rangle + 2\sqrt{2}\,|0,-1\rangle + \sqrt{2}\,|-1,0\rangle \right\}$$

$$= \frac{1}{\sqrt{3}}\left\{ 3\,|0,-1\rangle + 3\,|-1,0\rangle \right\} = \sqrt{3}\left\{ |0,-1\rangle + |-1,0\rangle \right\}$$

$$\Rightarrow \quad \boxed{\underset{j\ \ m}{|2,-1\rangle} = \frac{1}{\sqrt{2}}\left(\underset{m_1\ m_2}{|0,-1\rangle} + \underset{m_1\ m_2}{|-1,0\rangle} \right)} \quad (24)$$

and finally $\quad \boxed{\underset{j\ \ m}{|2,-2\rangle} = \underset{m_1\ m_2}{|-1,-1\rangle}} \quad (25)$

Now look at the $j=1$ multiplet. We start with (20)

$$\underset{j\ m}{|1,1\rangle} = \frac{1}{\sqrt{2}}\Big(\underset{m_1\ m_2}{|1,0\rangle} - \underset{m_1\ m_2}{|0,1\rangle}\Big)$$

Apply J_-. On the LHS

$$\boxed{\frac{1}{\hbar}J_-\underset{j\ m}{|1,1\rangle} = \underset{j(j+1)-m(m-1)}{\sqrt{1(1+1)-1(1-1)}}\ \underset{j,m-1}{|1,0\rangle} = \sqrt{2}\,|1,0\rangle} \tag{26}$$

On the RHS

$$\boxed{\begin{aligned}
&\frac{1}{\hbar}\left(J_{1-}+J_{2-}\right)\frac{1}{\sqrt{2}}\Big(\underset{m_1\ m_2}{|1,0\rangle} - \underset{m_1\ m_2}{|0,1\rangle}\Big) = \\
&\frac{1}{\hbar}\frac{1}{\sqrt{2}}\left\{J_{1-}|1,0\rangle - J_{1-}|0,1\rangle + J_{2-}|1,0\rangle - J_{2-}|0,1\rangle\right\} \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|0,0\rangle - \sqrt{2}\,|-1,1\rangle + \sqrt{2}\,|1,-1\rangle - \sqrt{2}\,|0,0\rangle\right\} \\
&= \underset{m_1\ m_2}{|1,-1\rangle} - \underset{m_1\ m_2}{|-1,1\rangle}
\end{aligned}} \tag{27}$$

$$\Rightarrow \quad \boxed{\underset{j\ m}{|1,0\rangle} = \frac{1}{\sqrt{2}}\Big(\underset{m_1\ m_2}{|1,-1\rangle} - \underset{m_1\ m_2}{|-1,1\rangle}\Big)} \tag{28}$$

We see immediately that this is orthogonal to $|2,0\rangle$ Eq (23), as it should be.

However, there are 3 distinct tensor product states with $m_1+m_2=0$. They are $\underset{m_1 m_2}{|1,-1\rangle}$, $\underset{m_1\ m_2}{|0,0\rangle}$ and $\underset{m_1 m_2}{|-1,1\rangle}$.

So far we have 2 orthogonal linear combinations corresponding to $\underset{j\ m}{|2,0\rangle}$ and

$\underset{j\ m}{|1,0\rangle}$. There must be a third linear combination orthogonal to both $\underset{j\ m}{|2,0\rangle}$ & $\underset{j\ m}{|1,0\rangle}$, which must correspond to $j=0, m=0$. Suppose this is

$$\boxed{\alpha|1,-1\rangle + \beta|0,0\rangle + \gamma|-1,1\rangle} \quad (29)$$

Orthogonality with $\underset{j\ m}{|2,0\rangle}$ means

$$\boxed{\alpha + 2\beta + \gamma = 0} \quad (30)$$

Orthogonality with $\underset{j\ m}{|1,0\rangle}$ means

$$\boxed{\alpha - \gamma = 0} \ (31) \implies \boxed{\alpha = \gamma \quad \text{and} \quad \beta = -\alpha} \quad (32)$$

Normalization implies $\alpha = \frac{1}{\sqrt{3}}$ up to a phase.

In the Condon-Shortley convention, we always choose the coefficient of the state with the largest value of m_1 in the linear combination to be positive. So

$$\boxed{\alpha = \frac{1}{\sqrt{3}} = \gamma = -\beta} \quad (33)$$

$$\implies \boxed{\underset{j\ m}{|0,0\rangle} = \frac{1}{\sqrt{3}}\left(\underset{m_1 m_2}{|1,-1\rangle} - \underset{m_1 m_2}{|0,0\rangle} + \underset{m_1 m_2}{|-1,1\rangle}\right)} \quad (34)$$

We go back and find $|1,-1\rangle$ by applying J_- to

$$\underset{j\ m}{J_- |1,0\rangle} = \sqrt{2}\, \underset{j\ m}{|1,-1\rangle} \quad \text{on the LHS}$$

$$(J_{1-} + J_{2-}) \frac{1}{\sqrt{2}} \left(\underset{m_1\ m_2}{|1,-1\rangle} - \underset{m_1\ m_2}{|-1,1\rangle} \right) = |0,-1\rangle - |-1,0\rangle$$

$$\Rightarrow \quad \boxed{\underset{j,\ m}{|1,-1\rangle} = \frac{1}{\sqrt{2}} \left(\underset{m_1\ m_2}{|0,-1\rangle} - \underset{m_1\ m_2}{|-1,0\rangle} \right)} \quad (35)$$

You can easily extend this procedure to any j_1, j_2.

This merely scratches the surface of what is known about the addition of angular momenta. One can ask about adding not just two, but three or more angular momenta, etc.