Problem 1 [20 points]: Electrostatics
Consider a sphere of radius $R$ with a spherically symmetric charge density which varies with $r$ as $\rho(r) = N r^{-1}$, where $N$ is some constant. The total charge of the sphere is $Q > 0$.
(a) Calculate $N$. [2 points]
(b) Calculate the electric field $\vec{E}$ both inside and outside the sphere. [10 points]
(c) Make a rough sketch of $|\vec{E}|$ as a function of $r$. [3 points]
(d) Calculate the work that would be required (i.e., against the action of the $\vec{E}$ field) to move a point charge $Q' > 0$ from $r = \infty$ to $r = 0$. [5 points]

Problem 2 [30 points]: Magnetic Force and Energy
As shown below, a rectangular loop of wire carrying a constant current $I_1$ is placed near an infinitely long wire carrying a constant current $I_2$. The rectangular loop is centered at $(d, 0, 0)$ in the indicated coordinate system. Note that the direction of the current $I_1$ in the wire is the same as the direction of the current $I_2$ in the side of the rectangular loop which is closest to the wire. Assume both $I_1$ and $I_2$ have been constant since $t = -\infty$.
(a) Calculate the magnetic vector potential $\vec{A}_2(\vec{x})$ of the long wire starting from the definition of $\vec{A}$ as given in Eq. (5.32) of Jackson. Hint: In order to most efficiently next work part (b), you might consider writing any integrals of the form as
\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du \: f(u) = 2 \int_{0}^{+\infty} du \: f(u)
\]
if the symmetry of the integrand permits you to do so. Then, you might consider writing any integrals of the form as
\[
\int_{0}^{+\infty} du \: f(u) = \lim_{L \to \infty} \int_{0}^{L} du \: f(u)
\]
and then proceed to evaluate the integral in terms of $L$. [10 points]
(b) Show that $\vec{B}_2(\vec{x}) = \hat{\nabla} \times \vec{A}_2(\vec{x})$ in the limit of $L \to \infty$ is as you would expect. [5 points]
(c) Calculate the two currents’ magnetic interaction energy in the limit of $L \to \infty$. [15 points]
Problem 3 [50 points]: Various Topics
Consider a solid conducting sphere of radius $R$ with a total integral charge of zero. The sphere rotates with constant angular velocity $\omega$ about the $z$-axis, so that $\vec{\omega} = \omega \hat{z}$. We will assume $\omega R \ll c$ (i.e., so that the velocity everywhere in the sphere is $\ll c$, which implies we can employ Galilean transformations). Suppose a magnetic dipole moment $\vec{m} = m \hat{z}$ is located at the center of the sphere, and is fixed in position (i.e., it does not move).

(a) Write an expression for the $\vec{B}$ field of the magnetic dipole moment $\vec{m}$ in terms of spherical coordinates $(r, \theta, \phi)$ and spherical unit vectors ($\hat{r}, \hat{\theta}, \hat{\phi}$) for $r > 0$. You do not need to derive it from first principles; it is sufficient to identify the appropriate expression in Jackson and then express it in spherical coordinates and spherical unit vectors. Give a physical argument for what the $\vec{B}'$ field in the sphere rest frame (i.e., in the reference frame co-rotating with the sphere) is in terms of the $\vec{B}$ field in the lab frame. [5 points]

(b) What is the electric field $\vec{E}'$ in the sphere rest frame (i.e., in the reference frame co-rotating with the sphere)? [3 points]

(c) Calculate the electric field $\vec{E}$ in the lab frame for $0 < r < R$. [7 points]

(d) What is the value of $\vec{\nabla} \times \vec{E}$ everywhere in space? [Hint: Think before you attempt any type of calculation.] Can $\vec{E}$ then, in principle, be written as $\vec{E} = -\nabla \Phi$, where $\Phi$ is the electrostatic potential? You do not need to find $\Phi$ here; a conceptual answer is sufficient. [5 points]

(e) Calculate the voltage (i.e., the potential difference) between some point $(r = R, \theta > 0)$ on the surface of the sphere and its North Pole. [10 points]

$$\text{Voltage} = \Phi(r = R, \theta > 0) - \Phi(r = R, \theta = 0)$$

(f) Find the charge density $\rho(r, \theta)$ which is established inside of the sphere for $0 < r < R$. Optional: You can check your answer by calculating the total integral charge. [5 points]

(g) Find the potential $\Phi(r, \theta)$ outside of the sphere for $r > R$. [15 points]

Potentially Useful Formulas

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left( u + \sqrt{u^2 + a^2} \right)$$

$$\hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$$

$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

$$P_0(\cos \theta) = 1 \quad P_1(\cos \theta) = \cos \theta \quad P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}$$

$$\sin^2 \theta = \frac{2}{3} P_0(\cos \theta) - \frac{2}{3} P_2(\cos \theta)$$

End of Exam