\[ \omega = \pm n = (re^{i\theta})^n = r^n e^{in\theta} \]

Choose \( V \) as pointed with \( n = 2 \).

\[
\begin{align*}
\theta = 0 & \Rightarrow V = r^n \sin(2\theta) = 0 \quad [\cos \theta = 0 \text{ in } \omega \text{ plane}] \\
\theta = \pi/2 & \Rightarrow V = r^n \sin(2\pi) = 0 \quad [\cos \theta = 0 \text{ in } \omega \text{ plane}] \\
\end{align*}
\]

\[ \phi = \frac{\theta + \pi}{2} \]

Then we have: (for \( n = 2 \))

\[
\left\{ \begin{array}{l}
V = r^2 \sin 2\theta \\
u = r^2 \cos 2\theta
\end{array} \right.
\]

Another example: \( \omega = \cos^{-1} z = \cos^{-1}(x+iy) \)

\[
\Rightarrow x + iy = \cos(u + iv) = \cos u \cosh v - i \sin u \sinh v
\]

\[
\left\{ \begin{array}{l}
x = \cos u \cosh v \\
y = -\sin u \sinh v
\end{array} \right. \Rightarrow \frac{x}{\cosh v} = \cos u \quad \cosh v = \frac{x}{\cos u} \\
-\frac{y}{\sinh v} = \sin u \quad \sinh v = \frac{-y}{\sin u}
\]

\[
\sin^2 u + \cos^2 u = 1 \Rightarrow \frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1
\]

\[
\cosh^2 v - \sinh^2 v = 1 \Rightarrow \frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = 1
\]
We can solve many complex problems with this. (Examples include):

1. If \( V \) represents a potential,
   - Field \( E \) between two elliptical cylinders (1)
   - Field external to a helical elliptical cylinder (2)
   - Etc.

2. If \( u \) represents the potential:
   - Field between two hyperbolic cylinders (1)
   - Two charged plates separated by a gap (V)

\[ \Phi = V_0 - V \]

So let \( u \) represent the potential \( \Phi \)!
Schwarz Transformation: Real Problem, Not (Easier) Inverse

We now treat the function (transformation) \( w = z^{-1} \) mapped an angle \( \gamma \) ranging the \( z \)-plane into the upper half of the \( w \)-plane, bounded by semi-infinite lines.

More generally, Schwarz transformation maps interior \( \gamma \) polygons in the \( z \)-plane to the upper half of the \( w \)-plane.

Let \( a_1 \) be fixed point in real \( \Re \), let \( z = g(w) \) be a function whose derivative is given by:

\[
\frac{dz}{dw} = (w - a_1)^{\gamma} \quad \text{for real } \Re, \quad -1 < \Re < 1
\]

\[
\frac{dz}{dw} \quad \text{exists and } i \theta \neq 0 \Rightarrow \text{transformation analytic}
\]

\[
-\frac{\pi}{\gamma} < \arg w < \frac{\pi}{\gamma}
\]

\[
\Rightarrow z(w) \text{ maps the interval } (a_1, \infty) \text{ onto a portion of a straight line}
\]

\[
dz = \Re e^{i\varphi} \cdot dw
\]

Similarly, if

\[
\arg \left[ z(w) \right] = \arg \left[ (w - a_1)^{\gamma} \right] = \Re e^{i\varphi} \text{ for } \Re > 0
\]

Generalizing, if have:

\[
z^{-1}(w) = A(w - a_1)^{\gamma}, \quad A \text{ complex constant, } \not= 0
\]

\[
\arg \left[ z^{-1}(w) \right] = \arg \left[ A(w - a_1)^{\gamma} \right] = \arg A + \Re \arg (w - a_1) \quad \text{for } \Re > 0
\]
Branch Cuts:

Consider \( z = r e^{i\theta} \)

\[ \Rightarrow \ln z = \ln r + i\theta \]

**Note:** if \( \theta \to \theta + 2\pi \), \( z \) does not change.

**But!** \( \ln z \) will change \( \Rightarrow \) multi-valued function!

\[ \ln z = \ln r + i(\theta + 2n\pi) \quad n = 0, 1, 2, \ldots \]

To avoid ambiguity, simplest choice is \( n = 0 \), limitation to interval of length \( 2\pi \),

such as \((-\pi, \pi)\).

Line in the \( z \)-plane that is not crossed, labelled a cut line or branch cut.

\[ \text{value of } \ln z \text{ with } n = 0 \text{ term } \text{"principal value of } \ln z \text{"}. \]