In 3D dimension:
\[ S(x-x') = S(y-y') S(z-z') \]
\[ \int V \int (x-x') dx' = \left\{ \begin{array}{ll}
1 & \text{if } V \text{ contains } x = x' \\
0 & \text{if not}
\end{array} \right. \]

Note: Delta function is dimension of an inverse volume in whatever dimensions the space has.

Next Time: start electronics, chapter 4

Lecture #2: Elements of: time-independent distribution of charge and fields

Recall from undergraduate E&M:

Coulomb's Law: The force \( F \) on a point charge \( q_1 \) located at \( x_i \) due to another
point charge \( q_2 \) located at \( x_j \) is:

\[ F = \frac{k q_1 q_2}{|x_i - x_j|^2} \]

In SI units, \( k = \frac{1}{4\pi \varepsilon_0} \)

Electric Field:

\[ E = \frac{q}{4\pi \varepsilon_0 r^2} \]

Field due to a point charge at origin:

\[ E_0 = 8.85 \times 10^{-12} \, \text{N} \cdot \text{m}^{-2} \text{C}^{-2} \]

Free space:

\[ q_1, q_2 \quad \text{in Coulombs} \quad (\text{C}) \quad x_i, x_j \quad \text{position} \quad E_1, E_2 \quad \text{vector} \quad \text{N} \quad \text{attractive} \]

Note: \( \pm \) is high charge, charge against \( + \) electron = \( 1.602 \times 10^{-19} \, \text{C} \)

Now, the electric field \( E \) can be defined as the force per unit charge acting at a given point.

\[ E = \frac{F}{q} \quad \Rightarrow \quad \text{Electric field at point } x \]

\[ E = \frac{k q_2}{|x - x_2|^2} \quad \text{due to point charge } q_2 \]

Comparing at \( x_2 \), it is:

Coulomb's law
At \( \vec{x} = \vec{x}' \), \( \vec{E}' = k_{\text{f}} \frac{\vec{x} - \vec{x}'}{|x_i - x_i'|^3} \).

Unite of charge field: \( \text{SI} \) \( \frac{\text{V}\cdot\text{m}}{\text{C}} \)

To act in vector:

chosen forces field \( \vec{F} \)

magnitude \( 1.44 \times 10^{-6} \text{ N} \) at

distance \( 1 \text{ m} \).

---

So far considered only one point charge. Generalize to a system of \( N \) point charges \( q_1, q_2, \ldots, q_N \) located at \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N \), by the principle of superposition:

\[
\vec{E}(\vec{x}) = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N} q_i \frac{\vec{x} - \vec{x}_i}{|x_i - x_i'|^3}
\]

If placed, \( \vec{F} \) point charges, we have a charge density \( \rho(\vec{x}') \) \( \text{SI} \) \( \frac{\text{C}}{\text{m}^3} \)

Discrete sum \( \rightarrow \) integral, \( dy' = \rho(\vec{x}') d^3x' \):

\[
\vec{E}(\vec{x}) = \frac{1}{4\pi \varepsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}_i}{|x_i - x_i'|^3} \ d^3x'
\]

This is the most general form. If we have a system of point charges, we consider the charge density at some point \( \vec{x}' \) as:

\[
\rho(\vec{x}') = \sum_{i=1}^{N} q_i \delta(\vec{x}' - \vec{x}_i)
\]

Integration

[\( dx' = dx'dy'dz' \) or the form for discrete cylindrical/spherical]
For a single point charge at position $x'$:

At arbitrary $x$, $\mathbf{E}$ is directed along $(x-x')$, depending on the sign of $q$.

$\Rightarrow$ the electric field from a point charge points radially outward/inward.

$E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$

$r$: vector from charge to point in question.
Taking the integral expression for $E$ we find:

\[
E(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i \delta(x^2 - x_i^2)(x-x_i)}{r^2} \Rightarrow \frac{1}{r} \text{: just doing variable of integration}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \left[ \frac{\delta(x-x_i)}{r} \int \frac{\delta(x^2 - x_i^2)(x-x_i)}{r^2} \, dx \right]
\]

\[
= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r^2} \frac{x-x_i}{r^2} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r^2} \frac{x-x_i}{|x-x_i|^3} \checkmark
\]

As we may recall from integral EM, calculating $E$ from general integral expression can be very difficult. Instead, Gauss's Law is sometimes more useful:

Suppose we have some surface $S$,

Consider charge inside the surface:

- $r$: distance from $q$ to surface
- $\hat{n}$: outwardly normal unit vector (1 to $S$)
- $\theta$: angle between $\vec{E}$ and $\hat{n}$

We can immediately write down for an infinitesimal surface area element $d\Omega$:

\[
E \cdot \hat{n} \, d\Omega = \left( \frac{1}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2} \right) \cdot \hat{n} \, d\Omega
\]

\[
= \frac{1}{4\pi\varepsilon_0} \frac{\hat{r} \cdot \hat{n}}{r^2} \, d\Omega
\]

\[
= r^2 \, d\Omega
\]

\[
\Rightarrow d\Omega = \frac{r^2}{r^2} \, d\Omega = \frac{d\Omega}{r^2}
\]

(recall: solid angle)

\[
\frac{\delta(r)}{r} \Rightarrow d\Omega = \frac{\delta(r)}{r^2}
\]
Solid Angle Subtended by Inifinitesimal Surface Area:

Surface \( S \)

\[ \hat{n} = \hat{r} \]

\[ d\Omega = \frac{da}{r^2} \]  \( (\text{not parallel to}) \)

What if \( \hat{n} \neq \hat{r} \)?

\[ d\Omega = \frac{da}{r^2} \]

\[ = \frac{da \cos \theta}{r^2} \]

\[ = \frac{da \cdot \hat{n} \cdot \hat{r}}{r^2} \]
\[ \int_E \hat{E} \cdot \hat{n} \, da = \frac{1}{4\pi \varepsilon_0} \int V \, d\Omega = \frac{Q}{\varepsilon_0} \]

If \( q \) is not inside the closed surface:

\[ \int_S \hat{E} \cdot \hat{n} \, da = 0 \]

[Every field line that enters the surface must also exit the surface \( \Rightarrow \text{net flux} = 0 \)]]

Generalizing:

\[ \int_S \hat{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \sum q_i \]

(for discrete set of point charges)

For continuous charge density \( \rho(x) \):

\[ \int_S \hat{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \int_V \rho(x) \, d^3x \]

[Integral form of Gauss's Law]

(value canceled by \( S \))

**Example:** Aperture, radius \( a \) has a spherically symmetric charge density \( \rho(r) = \frac{N}{r^2} \) centered at \( r = 0 \). Find \( \hat{E} \) inside and outside the sphere.

Use Gauss's Law:

\[ \int_S \hat{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \int_V \rho(x) \, d^3x \]

**Solution:**

1. Find \( \rho(x) \):

\[ Q = \int_V \frac{N}{r^2} \, d^3x \]

\[ Q = \int_0^a \int_0^{2\pi} \int_0^\pi \frac{N}{r^2} \, r^2 \sin \theta \, d\theta \, d\phi \, dr \]

\[ Q = \frac{4\pi N a^2}{2} \]

\[ \Rightarrow N = \frac{Q}{4\pi a^2} \]

**Total charge \( Q \):**

\[ \int_V \rho(v) \, d^3v = \frac{N}{v^2} \]

\[ N \text{ normal to } \hat{n} \]

\[ \rho(v) = \frac{Q}{4\pi a^2} \]
3. Find $E$: by symmetry, $E = E(\mathbf{r}) \hat{r}$, $\hat{n} = \hat{r}$ for sphere $r < a$.

\[\int_S E \hat{n} \, d\mathbf{a} = \int_S \hat{n} \cdot \mathbf{E} \, d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V E(\mathbf{r}) \, d^3x\]

\[\int_0^a \int_{\alpha=0}^{\pi} \int_0^{2\pi} E(\mathbf{r}) \, r^2 \sin \alpha \, dr \, d\phi \, d\alpha = \frac{1}{\varepsilon_0} \int_0^a \int_0^{2\pi} \int_0^{\pi} r^2 \sin \alpha \, \mathbf{E} \cdot \mathbf{r} \, r^2 \sin \alpha \, dr \, d\phi \, d\alpha\]

\[r^4 E(r) \cdot \bar{e}_r = (2\pi)(2\pi) \frac{Q}{4\pi \varepsilon_0 r} \]

\[E(r) = \frac{Q}{4\pi \varepsilon_0 r} \Rightarrow \vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \vec{r}
\]

For $r > a$:

\[E(r) \cdot \bar{e}_r = \frac{Q}{\varepsilon_0} \]

Recall Gauss's Theorem:

\[\int_0^a \mathbf{E} \cdot \hat{a} \, d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} \, d^3x = \frac{1}{\varepsilon_0} \int_V \mathbf{E} \cdot d^3x\]

\[\nabla \cdot \mathbf{E} = \frac{\mathbf{S}(\mathbf{r})}{\varepsilon_0}
\]

Contradiction at origin, $E = \frac{\mathbf{S}(\mathbf{r})}{\varepsilon_0}$.

\[\nabla \cdot \mathbf{E} = \frac{\varepsilon_0}{4\pi} \cdot \frac{\varepsilon_0}{4\pi} \Rightarrow p(\mathbf{r}) = \frac{\varepsilon_0}{4\pi} \cdot \frac{\varepsilon_0}{4\pi} \Rightarrow \mathbf{S}(\mathbf{r}) = \frac{\varepsilon_0}{4\pi} \mathbf{S}(\mathbf{r})
\]
Divergence of \( \frac{1}{r^2} \): \[ \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = \frac{1}{r^2} \nabla \cdot (r^2 \cdot \frac{\hat{r}}{r^2}) = \frac{1}{r^2} \cdot \sum_{r} (r) = \begin{cases} 0 & \text{for } r > 0 \\ 2 & \text{at } r = 0 \\ (\text{undefined}) & \end{cases} \]

By Gauss's Theorem: Consider, after
\[ \int_{\mathbb{V}} \nabla \cdot (r^2 \cdot \frac{\hat{r}}{r^2}) \, d^3x = \int_{\mathbb{S}} \frac{\hat{r}}{r^2} \cdot d^2S = \int_{\mathbb{S}} \sin \theta \, d\theta d\phi = 4\pi \]
\[ d^2S = r^2 \sin \phi \, d\phi d\phi \]

so must have \( \nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta(r) \) !

for divergence and to be satisfied.