Faraday's observations: \( E = -\frac{k}{dt} \) \( \Delta F \) \( \Delta t \) not an empirical constant

- Electric field in coordinate system frame for which all is at rest

\[ \oint \mathbf{E} \cdot d\mathbf{a} = -k \frac{dF}{dt} \]

\( k = 1 \) in SI, \( k = \frac{1}{c} \) in Gaussian

Galileo's law: a constant conducting wire of the circuit \( C \) in \( \mathbf{B} \)-field

implies relation between \( \mathbf{B} \) and \( \mathbf{E} \)-field nonetheless

\( \mathbf{E} \) in frame attached to a moving \( \mathbf{d} \) (i.e., the electric field in the frame where all is at rest, coordinate system moving at \( \mathbf{v} \) w.r.t. lab frame)
This result extended the circuit C and moving with (constant) velocity \( \vec{V} \).

Consider a second, identical circuit, \( C_2 \), which is at rest in the lab frame (i.e., \( \text{w.r.t.} \ C \) moving at \( \vec{V} \)).

Suppose that at some time the two circuits \( C \) and \( C_2 \) are (hypothetically) at exactly the same point in space (i.e., they overlap exactly).

In this circuit \( C_2 \) is at rest, Faraday's law will then be:

\[ \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} \quad \text{[NO transport terms!]} \]

from moving through non-uniform field!

By Galilean invariance, physics must be the same! (i.e., EMF must be the same, or equivalent current, must be the same at this instant in time)

\[ \vec{E}' = \vec{E} - k (\vec{V} \times \vec{B}) \]

-field in frame moving \( \vec{V} \)
-field in lab frame

\[ \vec{E}' = \vec{E} + \vec{V} \times \vec{B} \]  

[SI units]

To determine \( k \):

At this instant in time (on release):

To observe in lab frame, treat charge \( q \) in vacuum to electric field

\[ \vec{F} = q \vec{E} + q \vec{V} \times \vec{B} \]

\( \nabla \nabla \vec{E}' = k \vec{I} \text{ in SI} \]

Summary of Faraday’s law:

electric field in coordinate frame moving with velocity \( \vec{V} \) relative to lab frame

\[ \oint \vec{E}' \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} \quad \Rightarrow \text{changing magnetic flux generates electric field} \]

\[ \oint \left[ \vec{E} + (\vec{V} \times \vec{B}) \right] \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} + \int \left[ \vec{V} \times (\vec{V} \times \vec{B}) \right] \cdot d\vec{a} \]

\[ \oint \frac{\partial}{\partial t} \int \vec{E}' \cdot d\vec{a} \quad \Rightarrow \text{Time-dependent generalization of } \nabla \times \vec{E} = 0 \]

Differential form of Faraday's law.
Magnetic Field Energy

Assuming a linear relationship between $B$ and $H$ ($B = \mu_0 H$), the magnetic energy is:

$$\mathcal{W} = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{B} \, d^3x$$

Recall: energy in field is the work required to establish the field (starting from zero)

Note: if $\mathbf{H} > 0$, $\mathbf{B} = \mu_0 \mathbf{H}$ \quad $\Rightarrow$ \quad work done by current $\mathbf{J}$

Also, $\mathcal{W} = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{A} \, d^3x$ \quad interaction energy of current $\mathbf{J}$

\begin{align*}
\mathcal{W} & = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{A} \, d^3x \\
& = \frac{1}{2} \int \left( \mathbf{D} \times \mathbf{E} \right) \cdot d^3x + \frac{1}{2} \int \mathbf{B} \cdot \mathbf{B} \cdot d^3x \\
& = \frac{1}{2} \int \mathbf{E} \cdot d\mathbf{D} + \frac{1}{2} \int \mathbf{B} \cdot d\mathbf{B}
\end{align*}

Coefficient of Self- and Mutual-Inductance

Just like we defined coefficients of capacitance when we wrote potential energy of a system of $N$ capacitors, we write:

$$\mathcal{W} = \frac{1}{2} \sum_{i=1}^{N} \mathcal{C}_{ij} V_i V_j$$

Introducing coefficients of mutual and self-inductance for a system of $N$ distinct circuits, we:

$$\mathcal{W} = \frac{1}{2} \sum_{i=1}^{N} M_{ij} I_i I_j$$

because in general:

$$\mathcal{W} = \frac{1}{2} \int \mathbf{A} \times \mathbf{A} \cdot \mathbf{J} \, d^3x$$

with the integrals dependent on the geometry

\begin{align*}
\Rightarrow \quad \int_1 \text{ and } M_{ij} \text{ depend generally, on the geometry of the circuits.} & \\
\text{Indeed:} & \\
\mathcal{W} = \frac{1}{2} \int \mathbf{A} \times \mathbf{J} \cdot \mathbf{A} (\mathbf{r}) \quad \mathbf{A} = \frac{2\pi}{\varepsilon_0} \int \frac{\mathbf{J} (\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \\
& = \frac{1}{2} \int \mathbf{A} \times \mathbf{J} (\mathbf{r}) \cdot \frac{\mathbf{A} (\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mu_0}{2\pi} \int \mathbf{d}x \int \mathbf{d}x' \frac{\mathbf{J} (\mathbf{r}) \cdot \mathbf{J} (\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \quad |\mathbf{r} - \mathbf{r}'|