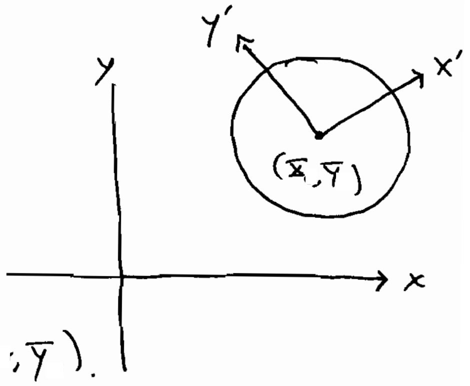
PHY 504 Problem Set #1 Solutions

1. (a) Let the frame of the person on the ground be centered at the origin, and the platform be centered at $(\bar{X}, \bar{Y})$ in the ground frame.



A point (x', y') on the platform will rotate with frequency ω about $(\bar{X}, \bar{Y})$.

The transformation is

$$x = x'\cos(\omega t) - y'\sin(\omega t) + \bar{X}$$

$$y = x'\sin(\omega t) + y'\cos(\omega t) + \bar{Y}$$

Inverting this gives

$$x' = (x - \bar{X})\cos(\omega t) + (y - \bar{Y})\sin(\omega t)$$

$$y' = -(x - \bar{X})\sin(\omega t) + (y - \bar{Y})\cos(\omega t)$$

(b) $\ddot{x} = \frac{d^2}{dt^2}(x'\cos\omega t - y'\sin\omega t) = \frac{d}{dt}(\dot{x}'\cos\omega t - \dot{y}'\sin\omega t - \omega(x'\sin\omega t + y'\cos\omega t))$

$$= \ddot{x}'\cos\omega t - \ddot{y}'\sin\omega t - 2\omega\dot{x}'\sin\omega t - 2\omega\dot{y}'\cos\omega t$$

$$-\omega^2(x'\cos\omega t - y'\sin\omega t) = 0$$

$$\ddot{y} = \ddot{x}'\sin\omega t + 2\dot{x}'\omega\cos\omega t - x'\omega^2\sin\omega t + \ddot{y}'\cos\omega t - 2\dot{y}'\omega\sin\omega t - y'\omega^2\cos\omega t = 0$$

To eliminate $\ddot{y}'$, multiply first equation by $\cos\omega t$, second by $\sin\omega t$, and add:

$$\boxed{\ddot{x}' - 2\omega\dot{y}' - \omega^2 x' = 0}$$

To eliminate $\ddot{x}'$, multiply first by $-\sin\omega t$, second by $\cos\omega t$ to get

$$\boxed{\ddot{y}' + 2\dot{x}'\omega - y'\omega^2 = 0}$$

The two extra terms in fact correspond to Coriolis and centrifugal acceleration.

(c) In the ground frame, $\boxed{\vec{x} = \vec{x}_0 + \vec{v}_0 t}$ since no forces act on the particle and $\ddot{\vec{x}} = 0$.

In the rotating frame,

$$x' = (x_0 + v_{0x}t - \bar{X})\cos\omega t + (y_0 + v_{0y}t - \bar{Y})\sin\omega t$$

$$y' = -(x_0 + v_{0x}t - \bar{X})\sin\omega t + (y_0 + v_{0y}t - \bar{Y})\cos\omega t$$

We can rewrite $\vec{x}_0, \vec{v}_0$ in terms of rotating coordinates:

$$x_0 = \bar{X} + x_0' \qquad v_{0x} = \dot{x}(0) = -x'\omega\sin\omega t - y'\omega\cos\omega t\Big|_{t=0} = -\omega y_0'$$

$$y_0 = \bar{Y} + y_0' \qquad v_{0y} = \dot{y}(0) = \omega x_0'$$

So finally
$$\boxed{\begin{aligned} x' &= (x_0' - \omega y_0' t)\cos\omega t + (y_0' + \omega x_0' t)\sin\omega t \\ y' &= -(x_0' - \omega y_0' t)\sin\omega t + (y_0' + \omega x_0' t)\cos\omega t \end{aligned}}$$

See 3(e) for a picture.

2. Let $m(t)$ = mass of rocket + fuel at time t.

In the inertial frame moving instantaneously with the rocket at time t, Newton's 2^{nd} Law gives

$$\sum_i F = F_{gas} - mg = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}v + m\frac{dv}{dt} = m\frac{dv}{dt}$$

where we have used the fact that $v(t) = 0$ in this frame.

F_{gas} is equal and opposite to the rate of change of the momentum of the escaping exhaust:

$$F_{gas} = -v'\frac{dm}{dt}$$

Note that $\frac{dm}{dt} \leq 0$ so $F_{gas} > 0$.

Putting together gives

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg .$$

Let m_0 = initial mass of rocket + fuel

m_e = mass of empty rocket.

Then $$m\frac{dv}{dm}\frac{dm}{dt} = -v'\frac{dm}{dt} - mg$$

$$\Rightarrow \frac{dv}{dm} = -\frac{v'}{m} - \frac{g}{dm/dt} = -\frac{v'}{m} - \frac{g}{\frac{1}{60}m_0}$$

$$\Rightarrow dv = \left(-\frac{v'}{m} - \frac{60g}{m_0}\right)dm$$

$$\Rightarrow v = -v'\log m - \frac{60g}{m_0}m \Big|_{m_0}^{m_e}$$

$$\Rightarrow \quad v = v' \log \frac{m_o}{m_e} - \frac{60g}{m_o}(m_e - m_o)$$

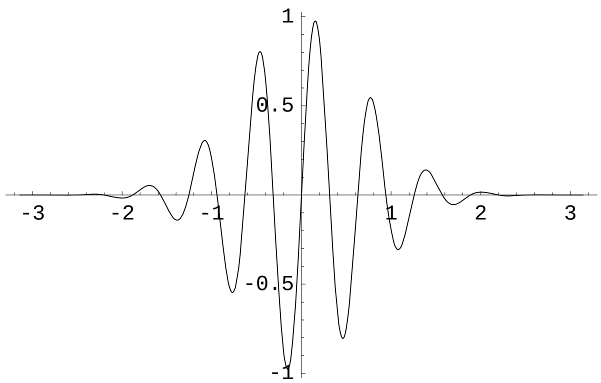
$$\approx v' \log \frac{m_f}{m_e} + 60g$$

where m_f = initial wt. of fuel and we used $m_o \gg m_e$.

Solve for

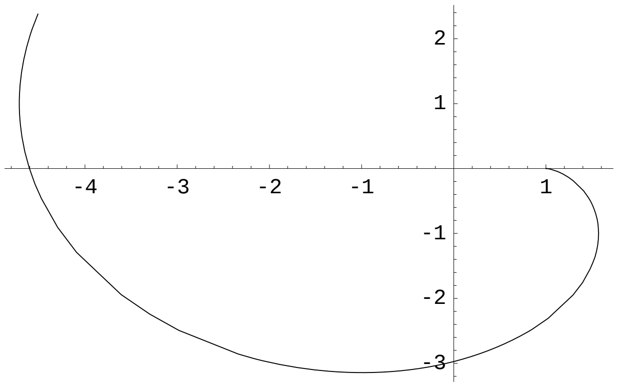
$$\frac{m_f}{m_e} = \exp \frac{v - 60g}{v'} = \exp \frac{11{,}200\,\mathrm{m/s} - 600\,\mathrm{m/s}}{2{,}100\,\mathrm{m/s}} \approx 274.$$

In[5]:= **Plot**[**Sin**[**10** * **x**] * **Exp**[-**x^2**], {**x**, -**Pi**, **Pi**}, **PlotRange** → **All**]



Out[5]= - Graphics -

In[6]:= **ParametricPlot**[{**Cos**[**t**] + **t** * **Sin**[**t**], -**Sin**[**t**] + **t** * **Cos**[**t**]}, {**t**, **0**, **5**}, **Axes** → **True**]



Out[6]= - Graphics -