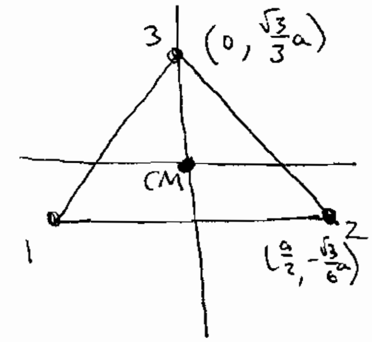
PHY 504 Problem Set #10 Solutions

1. Omitted. See example p. 253.



2. Arrange the masses as shown with the center of mass at the origin. Let $\vec{r}_{10}, \vec{r}_{20}, \vec{r}_{30}$ be the equilibrium positions, and let η_k and ζ_k be the deviations from equilibrium in the x- and y-directions, respectively.

Then

$$L = \frac{1}{2} m \left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2} + \dot{\vec{r}}_3^{\,2} \right) - \frac{1}{2} k \left[(r_{12} - a)^2 + (r_{23} - a)^2 + (r_{31} - a)^2 \right]$$

First look at the kinetic energy. Impose the CM constraint $\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = 0 \Rightarrow \dot{\vec{r}}_3 = -\dot{\vec{r}}_1 - \dot{\vec{r}}_2$:

$$T = \frac{1}{2} m \left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2} + (\dot{\vec{r}}_1 + \dot{\vec{r}}_2)^2 \right) = m \left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2} + \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 \right)$$

$$= m \left(\dot{\eta}_1^2 + \dot{\zeta}_1^2 + \dot{\eta}_2^2 + \dot{\zeta}_2^2 + \dot{\eta}_1 \dot{\eta}_2 + \dot{\zeta}_1 \dot{\zeta}_2 \right)$$

$$\equiv \frac{1}{2} T_{ij} \dot{q}_i \dot{q}_j$$

where

$$T_{ij} = m \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \zeta_1 \\ \zeta_2 \end{pmatrix}$$

Potential energy:

$$V = \frac{1}{2} k \left[\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - a \right)^2 + \text{cyclic} \right]$$

$$= \frac{1}{2} k \left[\left(\sqrt{(a + \eta_1 - \eta_2)^2 + (\xi_1 - \xi_2)^2} - a \right)^2 \right.$$

$$+ \left(\sqrt{\left(\frac{a}{2} + \eta_2 - \eta_3\right)^2 + \left(-\frac{\sqrt{3}}{2} + \xi_2 - \xi_3\right)^2} - a \right)^2$$

$$\left. + \left(\sqrt{\left(+\frac{a}{2} + \eta_3 - \eta_1\right)^2 + \left(\frac{\sqrt{3}}{2} + \xi_3 - \xi_1\right)^2} - a \right)^2 \right]$$

$$= \frac{1}{2} k a^2 \left[\left(\sqrt{\left(1 - \frac{\eta_1 - \eta_2}{a}\right)^2} - \cancel{a}\,1 \right)^2 \right.$$

$$+ \left(\sqrt{\left(\frac{1}{2} + \frac{\eta_2 - \eta_3}{a}\right)^2 + \left(-\frac{\sqrt{3}}{2} + \frac{\xi_2 - \xi_3}{a}\right)^2} - 1 \right)^2$$

$$\left. + \left(\sqrt{\left(\frac{1}{2} + \frac{\eta_3 - \eta_1}{a}\right)^2 + \left(\frac{\sqrt{3}}{2} + \frac{\xi_3 - \xi_1}{a}\right)^2} - 1 \right)^2 \right]$$

$$= \frac{1}{2} k a^2 \left[\left[- \frac{(\eta_1 - \eta_2)}{a} \right]^2 + \left[\frac{\eta_2 - \eta_3}{2a} - \sqrt{3}\, \frac{\xi_2 - \xi_3}{2a} \right]^2 \right.$$

$$\left. + \left[\frac{\eta_3 - \eta_1}{2a} + \sqrt{3}\, \frac{\xi_3 - \xi_1}{2a} \right]^2 \right]$$

$$= \frac{1}{2} k \left\{ \eta_1^2 - 2\eta_1\eta_2 + \eta_2^2 + \frac{1}{4}\left(\eta_2^2 - 2\eta_2\eta_3 + \eta_3^2\right) + \frac{3}{4}\left(\xi_2^2 - 2\xi_2\xi_3 + \xi_3^2\right) \right.$$

$$- \frac{\sqrt{3}}{2}\left(\eta_2\xi_2 - \eta_2\xi_3 - \eta_3\xi_2 + \eta_3\xi_3\right)$$

$$\left. \frac{1}{4}\left(\eta_3^2 - 2\eta_3\eta_1 + \eta_1^2\right) + \frac{3}{4}\left(\xi_3^2 - 2\xi_3\xi_1 + \xi_1^2\right) + \frac{\sqrt{3}}{2}\left(\eta_3\xi_3 - \eta_3\xi_1 - \eta_1\xi_3 + \eta_1\xi_1\right) \right\}$$

$$= \frac{1}{2} k \Big\{ \eta_1^2 - 2\eta_1\eta_2 + \eta_2^2$$

$$+ \frac{1}{4}\eta_1^2 + \eta_1\eta_2 + \eta_2^2 + 3\xi_1^2 + 12\xi_1\xi_2 + 12\xi_2^2 - 2\sqrt{3}\,\eta_1\xi_1 - 4\sqrt{3}\,\eta_1\xi_2 - 4\sqrt{3}\,\eta_2\xi_1 - 8\sqrt{3}\,\eta_2\xi_2$$

$$+ \eta_1^2 + \eta_1\eta_2 + \frac{1}{4}\eta_2^2 + 12\xi_1^2 + 12\xi_1\xi_2 + 3\xi_2^2 + 8\sqrt{3}\,\eta_1\xi_1 + 4\sqrt{3}\,\eta_1\xi_2 + 4\sqrt{3}\,\eta_2\xi_1 + 2\sqrt{3}\,\eta_2\xi_2 \Big\}$$

So $V = \frac{1}{2} V_{ij} q_i q_j$ where

$$V_{ij} = k \begin{pmatrix} \frac{9}{4} & 0 & \frac{3\sqrt{3}}{4} & 0 \\ 0 & \frac{9}{4} & 0 & -\frac{3\sqrt{3}}{4} \\ \frac{3\sqrt{3}}{4} & 0 & \frac{15}{4} & 3 \\ 0 & -\frac{3\sqrt{3}}{4} & 3 & \frac{15}{4} \end{pmatrix}$$

The equations of motion are

$$T_{ij}\ddot{q}_j = -V_{ij} q_j$$

If $q_j(t)$ is a normal mode of frequency ω, then $\ddot{q} = -\omega^2 q$,

so

$$-\omega^2 T_{ij} q_j = -V_{ij} q_j \quad \text{or} \quad (T^{-1}V)_{ij} q_j = \omega^2 q_i$$

In other words, the normal mode frequencies are the eigenvalues of $T^{-1}V$. T is block-diagonal, so it's easy to invert: just invert the 2×2 blocks to get

$$T^{-1} = \frac{1}{3m}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

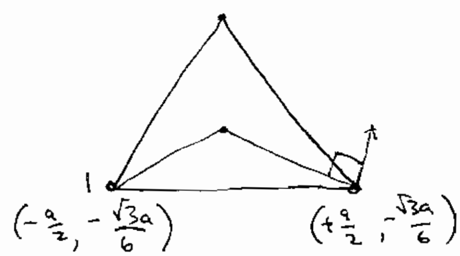
Then

$$T^{-1}V = \frac{k}{3m}\begin{pmatrix} \frac{9}{2} & -\frac{9}{4} & \frac{3\sqrt{3}}{2} & \frac{3\sqrt{3}}{4} \\ -\frac{9}{4} & \frac{9}{2} & -\frac{3\sqrt{3}}{4} & -\frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & \frac{3\sqrt{3}}{4} & \frac{9}{2} & \frac{9}{4} \\ -\frac{3\sqrt{3}}{4} & -\frac{3\sqrt{3}}{2} & \frac{9}{4} & \frac{9}{2} \end{pmatrix}$$

The eigenvalues of this matrix are (from Mathematica!)

$$\omega^2 = 0,\ \frac{3k}{m},\ \underbrace{\frac{3k}{2m},\ \frac{3k}{2m}}_{\text{double}}$$

The rotation mode about the origin is generated by the infinitesimal transformation



$$x \to x - \epsilon y$$
$$y \to y + \epsilon x$$

At vertex 1, this corresponds to $\eta_1 = -\epsilon y_{10} = +\epsilon \frac{\sqrt{3}}{6} a$, $\xi_1 = +\epsilon x_{10} = -\epsilon \frac{a}{2}$

At vertex 2, $\eta_2 = \epsilon \frac{\sqrt{3}}{6} a$ and $\xi_2 = +\epsilon \frac{a}{2}$.

The rotational mode is generated by the column vector $q_i = \frac{\epsilon a}{6}\begin{pmatrix} \sqrt{3} \\ +\sqrt{3} \\ -3 \\ 3 \end{pmatrix}$.

It is easily checked that this is an eigenvector of $T^{-1}V$ with eigenvalue 0.

Similarly, uniform stretching is generated by $q_i = \frac{\epsilon a}{6}\begin{pmatrix} 3 \\ -3 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$.

The corresponding eigenvalue is $\frac{3k}{m}$.