PHY 504 Problem Set #2 Solutions

1. (a) Let $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ = center of mass

$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ = relative position

$\vec{r}_1{}' = \vec{r}_1 - \vec{R}$ = center-of-mass coordinate of #1

$\vec{r}_2{}' = \vec{r}_2 - \vec{R}$

Then $\vec{r}_1{}' = \vec{r}_1 - \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \dfrac{m_2}{m_1+m_2}\vec{r}_{12}$

$\vec{r}_2{}' = -\dfrac{m_1}{m_1+m_2}\vec{r}_{12}$

Total kinetic energy $T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$

$$\tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} = \frac{m_1}{2}\left(\dot{\vec{R}} + \frac{m_2}{m_1+m_2}\dot{\vec{r}}_{12}\right)^2 = \tfrac{1}{2}m_1\dot{\vec{R}}^2 + \frac{m_1m_2^2}{2(m_1+m_2)^2}\dot{\vec{r}}_{12}^{\,2} + \frac{m_1m_2}{m_1+m_2}\dot{\vec{R}}\cdot\dot{\vec{r}}_{12}$$

$$\tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{m_2}{2}\left(\dot{\vec{R}} - \frac{m_1}{m_1+m_2}\dot{\vec{r}}_{12}\right)^2 = \tfrac{1}{2}m_2\dot{\vec{R}}^2 + \frac{m_1^2m_2}{2(m_1+m_2)^2}\dot{\vec{r}}_{12}^{\,2} - \frac{m_1m_2}{m_1+m_2}\dot{\vec{R}}\cdot\dot{\vec{r}}_{12}$$

$$\Rightarrow \quad T = \tfrac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}_{12}^{\,2} - \frac{m_1m_2}{m_1+m_2}\dot{\vec{R}}\cdot\dot{\vec{r}}_{12}$$

$$= \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}_{12}^{\,2} \quad \text{where } \mu = \frac{m_1m_2}{m_1+m_2}.$$

Angular momentum

$$\vec{L} = \vec{r}_1\times m_1\dot{\vec{r}}_1 + \vec{r}_2\times m_2\dot{\vec{r}}_2$$

$$= \left(\vec{R} + \frac{m_2}{m_1+m_2}\vec{r}_{12}\right)\times\left(m_1\dot{\vec{R}} + \frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}_{12}\right) + (1\leftrightarrow 2)$$

$$= \vec{R}\times(m_1\dot{\vec{R}}) + \frac{m_1m_2^2}{(m_1+m_2)^2}\vec{r}_{12}\times\dot{\vec{r}}_{12} + \frac{m_1m_2}{m_1+m_2}\vec{R}\times\dot{\vec{r}}_{12} + \frac{m_1m_2}{m_1+m_2}\vec{r}_{12}\times\dot{\vec{R}}$$

$$+ (1\leftrightarrow 2)$$

$$= \vec{R}\times M\dot{\vec{R}} + \vec{r}_{12}\times\mu\dot{\vec{r}}_{12}$$

(b) In the CM frame, $\vec{R} = \dot{\vec{R}} = 0$ so

$$T_{CM} = \frac{1}{2}\mu \dot{\vec{r}}_{12}^{\,2} \quad \text{and} \quad \vec{L}_{CM} = \vec{r}_{12} \times \mu \dot{\vec{r}}_{12}$$

If $\vec{L}_{CM}$ is in the $\hat{z}$-direction, then it is perpendicular to $\vec{r}_{12}$ and $\dot{\vec{r}}_{12}$, which therefore both lie in the xy-plane. Since $\vec{L}$ is conserved, this is true for all times, so the motion in the CM frame, as given by $\vec{r}_{12}(t)$, remains in the (xy)-plane.

In polar coordinates, $x_{12} = r\cos\theta$ and $y_{12} = r\sin\theta$, we have

$$T_{CM} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

and

$$L^z_{CM} = \mu\left(x_{12}\dot{y}_{12} - y_{12}\dot{x}_{12}\right)$$

$$= \mu\left(r\cos\theta\cdot(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) - r\sin\theta\,(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\right)$$

$$= \mu r^2\dot{\theta}$$

(d) For a rigid body, $r = \text{const.}$ so $\dot{r} = 0$.

Then $T_{CM} = \frac{1}{2}\mu r^2\dot{\theta}^2 = \dfrac{L^2}{2\mu r^2}$ and $E = \dfrac{L^2}{2\mu r^2} + V(r)$.

The motion will consist of the two particles rotating about their CM at constant $\dot{\theta}$, with the CM moving at constant velocity.

2.(a) Newton's 2^{nd} Law: $m\ddot{\vec{r}} = -\vec{\nabla} V$.

Convert this to polar coordinates:

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases} \Rightarrow \begin{array}{l} \ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \\ \ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \end{array}$$

Also

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial}{\partial x}\sqrt{x^2+y^2} + \frac{\partial V}{\partial \theta}\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{\partial V}{\partial r}\frac{x}{r} + \frac{\partial V}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial V}{\partial r}\cos\theta - \frac{\partial V}{\partial \theta}\frac{\sin\theta}{r}.$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\sin\theta + \frac{\partial V}{\partial \theta}\frac{\cos\theta}{r}.$$

Plug all this back into Newton:

(1) $$m\ddot{x} = m(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta) = -\frac{\partial V}{\partial r}\cos\theta + \frac{\partial V}{\partial \theta}\frac{\sin\theta}{r}$$

(2) $$m\ddot{y} = m(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta) = -\frac{\partial V}{\partial r}\sin\theta - \frac{\partial V}{\partial \theta}\frac{\cos\theta}{r}$$

Multiply (1) by $\cos\theta$, (2) by $\sin\theta$, and add:

$$\boxed{m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}}$$

Multiply (1) by $\sin\theta$, (2) by $-\cos\theta$, and add:

$$-2\dot{r}\dot{\theta} - r\ddot{\theta} = \frac{1}{r}\frac{\partial V}{\partial \theta} = 0 \quad (\text{since } V(r) \text{ is central})$$

or $\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$. This is equivalent

to conservation of L: $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$.

(b) Using results from problem 1,

$$E = T + V = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$L = m r^2 \dot{\theta}$$

The second equation implies $\dot{\theta} = \frac{L}{mr^2}$, so

$$E = \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \left(\frac{L}{mr^2} \right)^2 \right) + V = \tfrac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r).$$

(c) Just differentiate equations in (b) with respect to t.

(d)
$$E = \tfrac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{2mr^2} + V(r)$$

$$\Rightarrow \quad \frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - V(r)\right] - \frac{L^2}{m^2 r^2}}$$

$$\Rightarrow \quad dt = \frac{dr}{\sqrt{\frac{2}{m}(E - V(r)) - \frac{L^2}{m^2 r^2}}}$$

$$\Rightarrow \quad t(r) = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}(E - V) - \frac{L^2}{m^2 r'^2}}} + t_0$$

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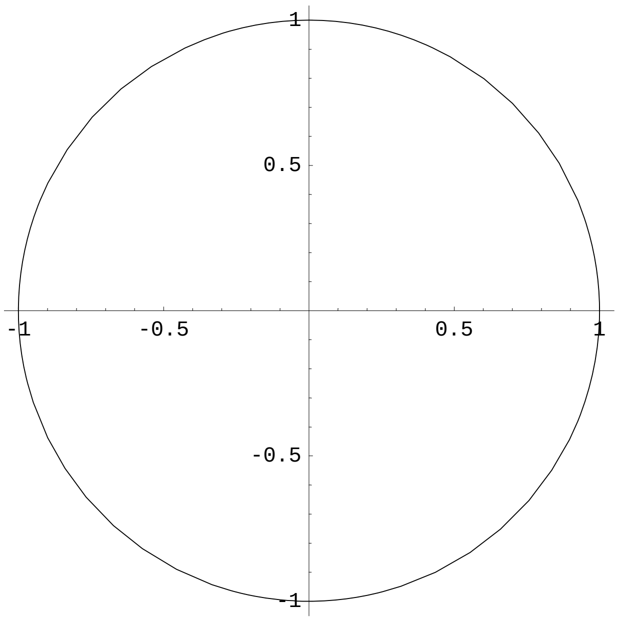
In[11]:= Clear[x, y, theta]
         x[theta_] = Cos[theta]
         y[theta_] = Sin[theta]
         ParametricPlot[{x[theta], y[theta]}, {theta, 0, 2*Pi}, AspectRatio → Automatic]

Out[12]= Cos[theta]

Out[13]= Sin[theta]
```

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Out[14]= - Graphics -
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3. (b) In 2(d) we found

$$t(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E-V(r)) - \frac{L^2}{m^2r^2}}} + t_0 = \int_{r_0}^{r} \frac{mr\,dr}{\sqrt{2mr^2(E-\frac{1}{2}(r-a)^2)-L^2}} + t_0$$

This will be integrable in terms of elementary functions only if $a=0$, which we assume.

Then substituting $u=r^2$,

$$t(r)-t_0 = \int_{r_0^2}^{r^2} \frac{m/2\,du}{\sqrt{2mu(E-\frac{u}{2})-L^2}} = \frac{\sqrt{m}}{2}\int_{r_0^2-E}^{r^2-E} \frac{dv}{\sqrt{(E^2-L^2/m)-v^2}}$$

$$= \frac{\sqrt{m}}{2}\sin^{-1}\left(\frac{v}{\sqrt{E^2-L^2/m}}\right)\Bigg|_{r_0^2-E}^{r^2-E}$$

or

$$t(r) = \frac{\sqrt{m}}{2}\sin^{-1}\frac{r^2-E}{\sqrt{E^2-L^2/m}}$$

where in the last line we chose t_0 to make the constant term vanish.

(c) See attached plot.

In[100]:=

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m = 1
L = 1
e = 2
t[r_] = Sqrt[m] / 2 * (ArcSin[(r^2 - e) / Sqrt[e^2 - L^2 / m]])
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Out[100]=

1

Out[101]=

1

Out[102]=

2

Out[103]=

$$\frac{1}{2}\ \text{ArcSin}\left[\frac{-2+r^2}{\sqrt{3}}\right]$$

In[117]:=

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Plot[t[r], {r, .6, 1.9}]
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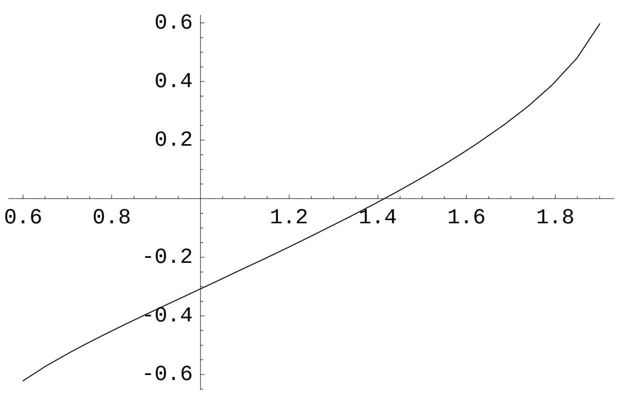

Out[117]=

- Graphics -