PHY 504 Problem Set #3 Solutions

1. Let $(q_1, \dots, q_n)$ be generalized coordinates,

and

$$q_i = q_i(s_1, \dots, s_n, t) \qquad i = 1, \dots, n$$

a coordinate transformation.

Define

$$L(s_1, \dots, s_n; t) = L(q_1(s_i), q_2(s_i), \dots; t)$$

We wish to show that if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_i} = 0$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

Now

$$\dot{q}_i = \sum_{j=1}^{n} \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}$$

so

$$\frac{\partial \dot{q}_i}{\partial \dot{s}_j} = \frac{\partial q_i}{\partial s_j}.$$

Also

$$\frac{\partial L}{\partial s_j} = \sum \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j}$$

$$\frac{\partial L}{\partial \dot{s}_j} = \sum \frac{\partial L}{\partial q_i}\underbrace{\cancel{\frac{\partial q_i}{\partial \dot{s}_j}}}_{0} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}$$

$$\therefore \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j} = \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial q_i}{\partial s_j}\right)$$

$$= \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j} = \frac{\partial L}{\partial s_j}$$

Q.E.D. //

2. (a) Let the length of the pendulum be l. Then

$$L = \frac{1}{2} m v^2 + V = \frac{1}{2} m (l \dot{\theta})^2 + mgl \cos\theta$$

Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta}\right) = -mgl \sin\theta$$

$$\text{or} \quad \ddot{\theta} = -\frac{g}{l} \sin\theta .$$

(b) For small displacements from $\theta = 0$, $\sin\theta \approx \theta$ and

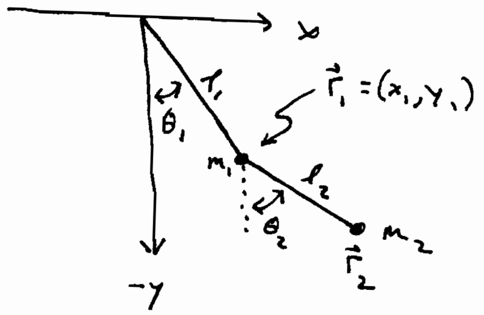
$$\ddot{\theta} \approx -\frac{g}{l} \theta$$

This is an oscillator of frequency $\omega = \sqrt{\frac{g}{l}}$ with general solution

$$\theta(t) = A \cos \sqrt{\frac{g}{l}}(t - t_0)$$

(c) Just copy the code into Mathematica!

3(a) As shown in the figure we let $\vec{r}_1$ and $\vec{r}_2$ be the positions of the two masses.



The Lagrangian is

$$L = T - V = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - m_1 g y_1 - m_2 g y_2$$

and the rigid-rod constraint fixes

$$|\vec{r}_1| = l_1 \quad \text{and} \quad |\vec{r}_2 - \vec{r}_1| = l_2$$

The constraints can be solved by choosing the generalized coordinates (θ_1, θ_2) where

$$x_1 = l_1 \sin\theta_1 \qquad\qquad x_2 - x_1 = l_2 \sin\theta_2$$

and

$$y_1 = -l_1 \cos\theta_1 \qquad\qquad y_2 - y_1 = l_2 \cos\theta_2$$

(b) Then

$$L = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) \right]$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

The Lagrange equation for θ_1:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \frac{d}{dt}\left(m_1 l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2) \right)$$
$$= (m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \left[\ddot\theta_2 \cos(\theta_1 - \theta_2) - \dot\theta_2 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1 - \theta_2) \right]$$

$$\frac{\partial L}{\partial \theta_1} = \pm l_1 g (m_1 + m_2) \sin\theta_1 - m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2)$$

Putting these together gives

$$(m_1+m_2)\,l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0$$

Similarly

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 g \sin\theta_2 = 0$$

(c) For small angles, $\sin\theta \approx \theta$ and $\cos\theta \approx 1-\frac{\theta^2}{2}$.

Inserting into the above equations of motion and keeping only terms linear in θ:

$$\begin{cases} (m_1+m_2)\,l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1+m_2) g \,\theta_1 = 0 \\ m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 - m_2 g \theta_2 = 0 \end{cases}$$

4. (a) $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0 \quad \Rightarrow \quad \ddot{x} = -\frac{k}{m}x$$

$$\Rightarrow \quad x(t) = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$

(b) $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$.

Boundary conditions: $x(0) = x(T) = 0 \Rightarrow x(t) = B\sin\sqrt{\frac{k}{m}}\,t$.

Plug into action:

$$I = \int_0^T L\,dt = \int_0^T \frac{1}{2}(m\dot{x}^2 - kx^2)\,dt$$

$$= \frac{1}{2}B^2\int_0^T \left[m\omega^2\cos^2\omega t - k\sin^2\omega t\right]dt$$

$$= \frac{1}{2}kB^2\int_0^T (\cos^2\omega t - \sin^2\omega t)\,dt$$

$$= \frac{1}{2}kB^2\int_0^T \cos 2\omega t\,dt = \frac{1}{2}kB^2 \left.\frac{\sin 2\omega t}{2\omega}\right|_0^T = 0$$

(c) Suppose $X(t) = \sum_{n=1}^{\infty} a_n \sin(n\omega t)$.

Then
$$\dot{x} = \sum n\omega a_n \cos(n\omega t)$$

and
$$I = \int_0^T L\,dt = \frac{1}{2}\int_0^T (m\dot{x}^2 - kx^2)\,dt$$

$$= \frac{1}{2}\int_0^T \left(m\omega^2\left[\sum n a_n \cos(n\omega t)\right]^2 - k\left[\sum a_n \sin(n\omega t)\right]^2\right) dt$$

$$= \frac{k}{2}\int_0^T \Big[\sum_n n^2 a_n^2 \cos^2(n\omega t) + \sum_{n\neq m} n a_n a_m \cos(n\omega t)\cos(m\omega t)$$
$$- \sum_n a_n^2 \sin^2(n\omega t)\Big]\,dt$$

$$= \frac{k}{2}\sum_n \left(n^2 a_n^2 \cdot \frac{T}{2} - a_n^2 \cdot \frac{T}{2}\right) dt$$

$$= \frac{kT}{4}\sum_n (n^2-1)a_n^2 = \frac{k\pi}{2\omega}\sum_{n=1}^{\infty}(n^2-1)a_n^2$$

(d) Because $a_n^2 > 0$ and $n \geq 1$, $I \geq 0$ for *any* path, and the path studied in (b) is a minimum.