PHY 504 Problem Set #4 Solutions

1. Let $q_i(t)$ describe the motion of a system with Lagrangian $L(q_i, \dot{q}_i, \ddot{q}_i; t)$, and consider a family of paths parameterized by α

$$q_1(t,\alpha) = q_1(t) + \alpha\eta_1(t)$$
$$q_2(t,\alpha) = q_2(t) + \alpha\eta_2(t)$$
$$\vdots$$

which reduce to $q_i(t)$ when $\alpha = 0$. Assume that $\eta_i(t_1) = \eta_i(t_2) = \dot{\eta}_i(t_1) = \dot{\eta}_i(t_2) = 0$.

Hamilton's principle states that, for any such $\eta_i(t)$, the variation of the action with respect to α vanishes at $\alpha = 0$:

$$0 = \left.\frac{\partial I}{\partial\alpha}\right|_{\alpha=0} = \frac{\partial}{\partial\alpha}\int_{t_1}^{t_2} L(q_1(t,\alpha) \ldots \dot{q}_1(t,\alpha) \ldots \ddot{q}_1(t,\alpha) \ldots; t)\, dt$$

$$= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial\alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial\alpha} + \frac{\partial L}{\partial \ddot{q}_i}\frac{\partial \ddot{q}_i}{\partial\alpha} \right) dt$$

$$= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i + \frac{\partial L}{\partial \ddot{q}_i}\ddot{\eta}_i \right) dt$$

As in Eq. (2.17), the second term may be integrated by parts

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{d\eta_i}{dt}\, dt = \cancelto{0}{\left.\frac{\partial L}{\partial \dot{q}_i}\eta_i\right|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i\, dt$$

The surface term vanishes because $\eta_i(t_1) = \eta_i(t_2) = 0$.

The third term may be integrated by parts twice

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \ddot{q}_i} \frac{d^2 \eta_i}{dt}\, dt = \cancelto{0}{\frac{\partial L}{\partial \ddot{q}_i} \dot{\eta}_i \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_i} \right) \frac{d\eta_i}{dt}\, dt$$

$$= \cancelto{0}{-\left(\frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_i} \right) \eta_i \Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2} \left(\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} \right) \eta_i\, dt$$

Putting these results together gives

$$0 = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} \right) \eta_i\, dt$$

Since this must be true for all η_i satisfying the boundary conditions, it follows as in Eq. (2.18) that the expression in parentheses must vanish for each i

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i} \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} = 0 \qquad \text{QED}$$

For $L = -\frac{m}{2} q \ddot{q} - \frac{k}{2} q^2$ we find

$$0 = \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = \frac{d^2}{dt^2}\left(-\frac{m}{2} q\right) - 0 + \left(-\frac{m}{2}\ddot{q} - kq\right)$$

$$= -m\ddot{q} - kq \qquad \text{or} \qquad m\ddot{q} = -kq .$$

This is the equation of motion of a harmonic oscillator, which also follows from $L_{osc} = \frac{m}{2}\dot{q}^2 - \frac{k}{2} q^2$. This differs from the given Lagrangian by a total t-derivative $L - L_{osc} = \frac{d}{dt}\Big(\frac{m}{2} q$ which in view of Deriv. 1.8 is why they have the same EOM.

2 (a) Start with the Lagrangian of a particle in a gravitational field

$$L_0 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

(we ignore motion in the z-direction, which is trivial)

and impose the constraint $y = kx^2$ using a Lagrange multiplier λ:

$$L = L_0 + \lambda(y - kx^2)$$

(b) Equations of motion:

for x: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + 2\lambda k x = 0$ (1)

for y: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + mg - \lambda = 0$ (2)

for λ: $y - kx^2 = 0$ (3)

Solving (2) for λ gives

$$\lambda = m\ddot{y} + mg = m\,\ddot{(kx^2)} + mg = m(2kx\ddot{x} + 2k\dot{x}^2) + mg$$

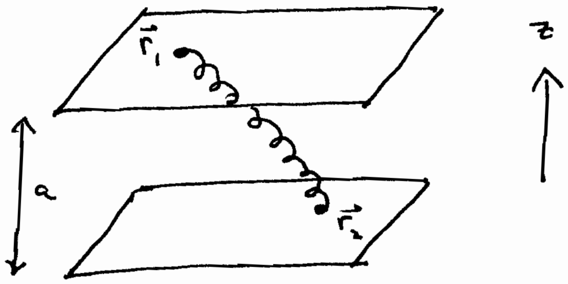
Plugging into (1) gives

$$m\ddot{x} + 2\left[2km(x\ddot{x} + \dot{x}^2) + mg\right]kx = 0$$

$$\Rightarrow \ddot{x} + 4k^2(x^2\ddot{x} + x\dot{x}^2) + 2gkx = 0$$

(c) For small x (and $\dot{x}$ and $\ddot{x}$) we can neglect the middle term, leaving $\ddot{x} = -2gkx$. This is a harmonic oscillator of frequency $\omega = \sqrt{2gk}$.

3. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of the two particles, and suppose that $\vec{r}_1$ moves on the plane $z_1 = -\frac{a}{2}$ and $\vec{r}_2$ is on the plane $z_2 = +\frac{a}{2}$.



(We make this choice to have symmetry under reflection about the xy-plane.)

Then the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}\right) - \frac{1}{2}k\left|\vec{r}_1 - \vec{r}_2\right|^2 + \lambda_1\left(z_1 + \frac{a}{2}\right) + \lambda_2\left(z_2 - \frac{a}{2}\right)$$

where we have included Lagrange multiplier terms to implement the constraints $z_1 = -\frac{a}{2}$ and $z_2 = +\frac{a}{2}$.

The constraints can be solved by simply setting z_1 and z_2 to these values in L:

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) - \frac{1}{2}k\left((x_1 - x_2)^2 + (y_1 - y_2)^2 + a^2\right)$$

This is also the Lagrangian for two independent pairs of particles, (x_1, x_2) and (y_1, y_2), in one dimension

The symmetries are most evident in center-of-mass coordinates $\vec{R} = (X, Y) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$, $\vec{s} = \vec{r}_1 - \vec{r}_2$

$$L = m\dot{\vec{R}}^2 + \frac{m}{4}\dot{\vec{s}}^2 - \frac{k}{2}\vec{s}^2 \qquad \text{(ignoring a constant).}$$

where $\vec{R}$ and $\vec{s}$ are 2d vectors.

The symmetries and conserved quantities are

(i) Translations of $\vec{R}$ in x- and y- directions $\rightarrow$ x- and y- components of center-of-mass momentum P_x and P_y are conserved, where $P_x = 2m\dot{X}$ and $P_y = 2m\dot{Y} = \frac{\partial L}{\partial \dot{Y}}$.

(ii) Rotations of $\vec{R} \rightarrow$ angular momentum of center-of-mass $\vec{R} \times \vec{P} = 2mR\dot{R}\hat{z}$ is conserved.

(iii) Rotations of $\vec{r} \rightarrow$ "internal" angular momentum about center-of-mass $\vec{r} \times \vec{P}_s = \frac{1}{2} m s \dot{s}\, \hat{z}$ conserved (where $p_x = \frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m \dot{s}$)

(iv) Time translations $\rightarrow$ conservation of energy

$$E = \vec{P} \cdot \dot{\vec{R}} + p \cdot \dot{\vec{s}} - L$$

$$= 2m\dot{\vec{R}}^2 + \frac{1}{2} m \dot{\vec{s}}^2 - L = m\dot{\vec{R}}^2 + \frac{1}{4} m \dot{\vec{s}}^2 + \frac{k}{2} \vec{s}^2 .$$

In terms of original coordinates

$$E = T + V = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} k\left((x_1 - x_2)^2 + (y_1 - y_2)^2 + a^2\right)$$

(v) Reflection about xy-plane. This is a discrete symmetry and has no associated conserved qty.

4. (a) $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda\left(\sqrt{x^2+y^2} - a\right)$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + \lambda(r - a)$$

Symmetries	Conserved quantities
z - translation	z-momentum p_z
(xy) rotation	L_z
t - translation	E

(b) $L = \frac{1}{2}m\left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}\right) + \frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|}$

$$= m\dot{\vec{R}}^2 + \frac{m}{4}\dot{s}^2 + \frac{Gm^2}{s}$$

Symmetries	
$\vec{R}$ - translations	Center-of-mass momentum
$\vec{R}$ - rotations	CM angular momentum
$\vec{s}$ - rotations	Internal angular momentum
t - translations	Energy.
Interchange ($\vec{r}_1 \leftrightarrow \vec{r}_2$)	—

(c) $L = \frac{1}{2}m\left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}\right) + \frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|} + \lambda_1(|\vec{r}_1| - a) + \lambda_2(|\vec{r}_2| - a)$

The constraints break $\vec{R}$ - translation symmetry compared to (b), but preserve overall rotational symmetry about the origin (in which $\vec{r}_1$ and $\vec{r}_2$ are simultaneously rotated). Total angular momentum is conserved. Independent rotations of r_1 and r_2 are not symmetries in general.

The system is also symmetric under t-translations and $\vec{r}_1 \leftrightarrow \vec{r}_2$.

(d) Suppose $\vec{E}$ is in the z-direction.

Then $$L = \frac{1}{2} m \dot{\vec{r}}^2 + qEz$$

(This can be derived from Eq. (1.63) with $\phi = -Ez$ so $\vec{E} = -\vec{\nabla}\phi = E\hat{z}$)

This is $\underline{\text{not}}$ symmetric under z - translations.

Symmetries	Conserved
x- and y- translations	$p_x = m\dot{x}$, $p_y = m\dot{y}$
(xy) - rotations	$L_z = m(x\dot{y} - y\dot{x})$
t - translation	E

(e) Suppose $\vec{B} = B\hat{z}$. Then take $\vec{A} = Bx\hat{y}$ so that $\vec{\nabla} \times \vec{A} = \vec{B}$.

$$L = \frac{1}{2} m \dot{\vec{r}}^2 + q\vec{A} \cdot \dot{\vec{r}} = \frac{1}{2} m \dot{\vec{r}}^2 + qBx\dot{y}$$

This is invariant under y- and z-translations. The associated conserved quantities are $p_z = m\dot{z}$ and $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qBx$.

The geometry suggests that there should also be a symmetry under x-translations; however, these do not leave L invariant:

$$x \to x + \epsilon \quad \text{gives} \quad L \to L + qB\epsilon\dot{y} \equiv L + \epsilon \Delta L$$

$$\text{where} \quad \Delta L = qB\dot{y} = \frac{d}{dt}(qBy) \equiv \dot{J}$$

Since L changes only by a total time derivative, x-translations are a symmetry, with conserved momentum

$$p_x = \frac{\partial L}{\partial \dot{x}} \Delta x - J = m\dot{x} - qBy$$

Note the similarity in form to p_y above.

(we could also have derived this by going to the alternative gauge $\vec{A} = -By\hat{x}$.)

We also expect rotations in the (xy)-plane to be a symmetry:

$$\begin{cases} x \to x + \epsilon y \\ y \to y - \epsilon x \end{cases} \quad \Rightarrow \quad L \to L + qB\epsilon y\dot{y} - qB\epsilon x\dot{x}$$

$$= L + \tfrac{1}{2}\epsilon qB \frac{d}{dt}(y^2 - x^2)$$

Conserved:

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x + \frac{\partial L}{\partial \dot{y}} \Delta y - J$$

$$= m\dot{x}y + (m\dot{y} + qBx)(-x) - \tfrac{1}{2} qB(y^2 - x^2)$$

$$= m(\dot{x}y - \dot{y}x) - \tfrac{1}{2} qB(x^2 + y^2)$$

Finally, energy is

$$E = \frac{\partial L}{\partial \dot{x}} \dot{x} + \frac{\partial L}{\partial \dot{y}} \dot{y} + \frac{\partial L}{\partial \dot{z}} \dot{z} - L$$

$$= m\dot{x}^2 + (m\dot{y} + qBx)\dot{y} + m\dot{z}^2 - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - qBx\dot{y}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$