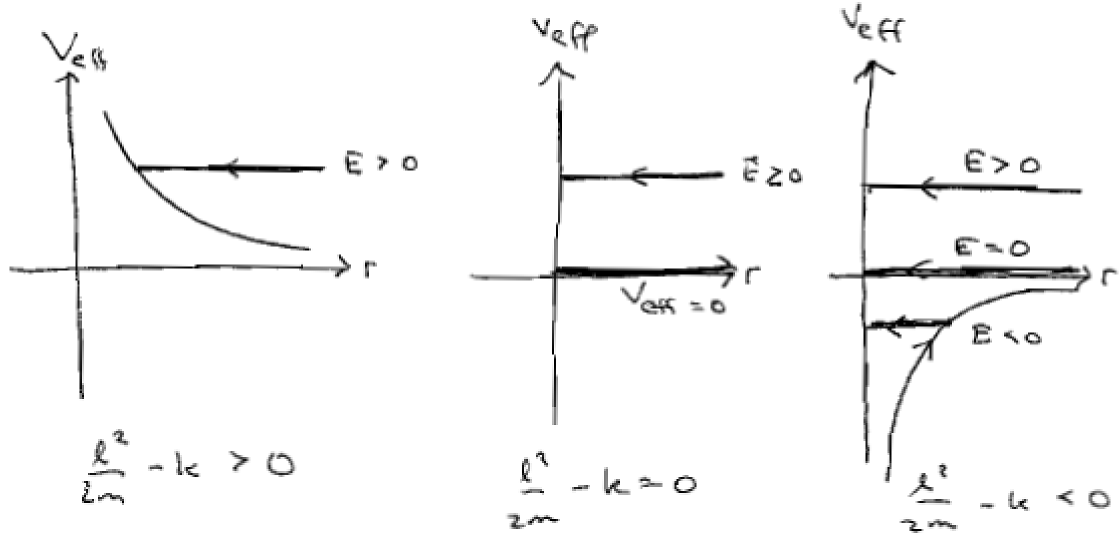
PHY504 Problem Set #5 Solutions

1. The effective potential is

$$V_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{k}{r^2} = \frac{\left(\frac{\ell^2}{2m} - k\right)}{r^2}$$

This can take 3 qualitative forms, depending on the sign of $\frac{\ell^2}{2m} - k$:



In the first two cases, there is one type of orbit for each case. In the third case $\frac{\ell^2}{2m} < k$ there are 3 distinct types of orbits; depending on whether $E > 0$, $E = 0$, or $E < 0$.

These 5 types of orbits correspond to the 5 orbits listed in the problem, as we will explain.

(a) & (b) From Eq. (3.39) with $n = -3$ we have

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} + \left(\frac{2mk}{l^2} - 1\right)u^2}} \qquad \left(V = -\frac{k}{r^2}\right)$$

We consider the following 5 cases, corresponding to the 5 orbits described above:

(i) $E > 0$ and $l^2 > 2mk$

$$\theta = \theta_0 - \sqrt{\frac{l^2}{l^2 - 2mk}} \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2 - 2mk} - u^2}}$$

$$\Rightarrow \quad \theta = \theta_0 - \sqrt{\frac{l^2}{l^2 - 2mk}} \cos^{-1}\left(\sqrt{\frac{l^2 - 2mk}{2mE}}\, u\right)\Bigg|_{u_0 = \sqrt{\frac{2mE}{l^2 - 2mk}}}^{u} \qquad (\text{see eq. } 3.51)$$

$$\therefore \quad u = \frac{1}{r} = \sqrt{\frac{2mE}{l^2 - 2mk}} \cos \sqrt{\frac{l^2 - 2mk}{l^2}}\,(\theta - \theta_0)$$

(ii) $E > 0$ and $l^2 = 2mk$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2}}} = \theta_0 - \sqrt{\frac{l^2}{2mE}}\,(u - u_0)$$

This case is a little tricky because $u_0 = \frac{1}{r_0} = \infty$, but θ is also winding $+\infty$ as $r \to 0$. Alternatively can take the reference point (u_0, θ_0) to be $(0, \theta_0)$, corresponding to a scattering orbit that starts (at $r = \infty, u = 0$) at angle $\theta = \theta_0$

Then $u = \sqrt{\frac{2mE}{l^2}}\,(\theta - \theta_0)$

(iii) $E > 0, \ \ell^2 < 2mk$:

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2}}\sqrt{1+\left(\sqrt{\frac{2mk-\ell^2}{2mE}}\,u\right)^2}} = \theta_0 - \frac{\sqrt{\ell^2}}{\sqrt{2mk-\ell^2}} \sinh^{-1}\left(\sqrt{\frac{2mk-\ell^2}{2mE}}\,u\right)$$

$$u = \frac{1}{r} = \sqrt{\frac{2mE}{2mk-\ell^2}} \sinh\left[-\sqrt{\frac{2mk-\ell^2}{\ell^2}}\,(\theta-\theta_0)\right]$$

(iv) $E = 0, \ \ell^2 < 2mk$:

$$u = \frac{1}{r} = \frac{1}{r_0} \exp\left[\pm\sqrt{\frac{2mk-\ell^2}{\ell^2}}\,(\theta-\theta_0)\right]$$

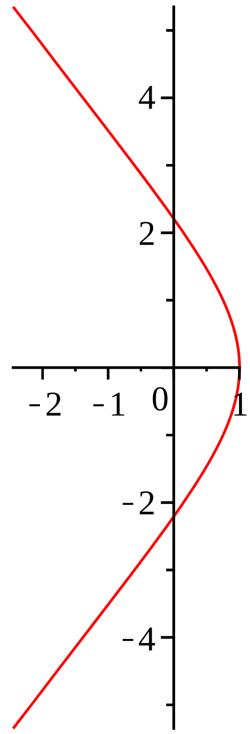
for a trajectory passing through the point (r_0, θ_0).

(v) $E < 0, \ \ell^2 < 2mk$:

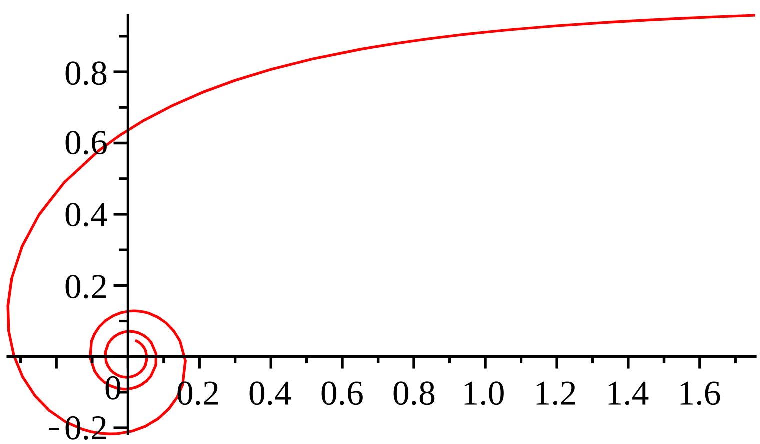
$$u = \frac{1}{r} = \sqrt{\frac{2m|E|}{2mk-\ell^2}} \cosh\left[\sqrt{\frac{2mk-\ell^2}{\ell^2}}\,(\theta-\theta_0)\right]$$

See attached Maple plots of these orbits.

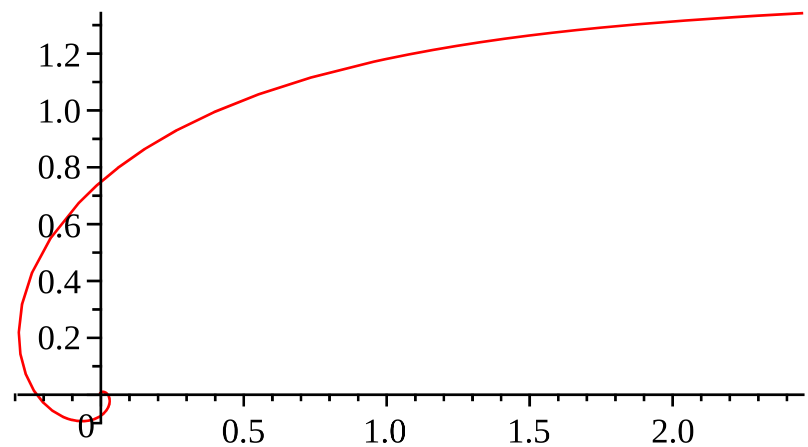
> $plot\left(\frac{1}{\cos(.7 \cdot t)}, t=-2..2, coords=\text{polar}, scaling=constrained\right);$



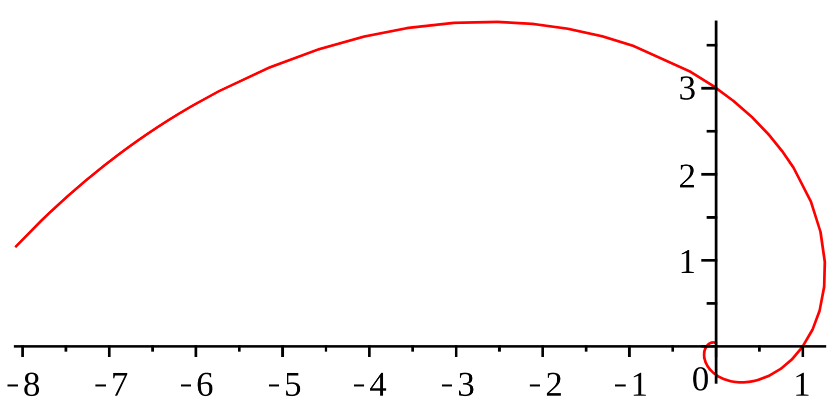
> $plot\left(\frac{1}{t}, t=0.5..20, coords=\text{polar}, scaling=constrained\right);$



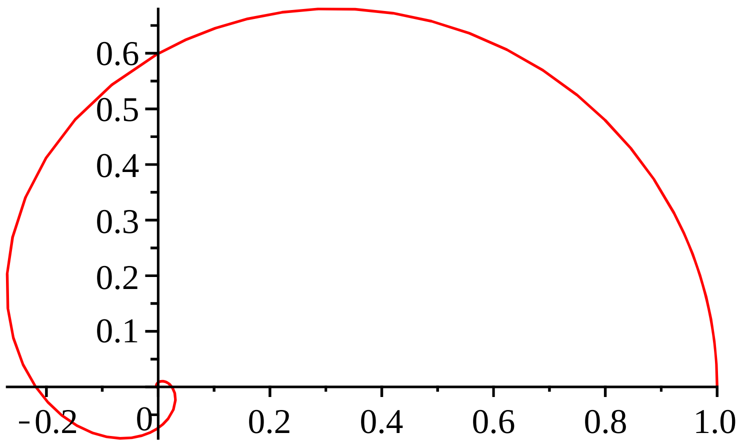
$> plot\left(\frac{1}{\sinh(.7\cdot t)}, t = 0.5..10, coords = \text{polar}, scaling = constrained\right);$



$> plot(\exp(.7\cdot t), t = -5..3, coords = \text{polar}, scaling = constrained);$



> $plot\left(\frac{1}{\cosh(.7\cdot t)}, t = 0..15, coords = \mathrm{polar}, scaling = constrained\right);$



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2. a) A circular orbit sits at the bottom of the effective potential

$$V_{eff}(r) = -ar^{n+1} + \frac{l^2}{2mr^2}$$

$$\frac{dV_{eff}}{dr} = -a(n+1)\,r_0^n - \frac{l^2}{mr^3} = 0 \Rightarrow r_0 = \left(\frac{-l^2}{a(n+1)m}\right)^{\frac{1}{n+3}}$$

To find the period, recall that area is swept out at a constant rate

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{l}{2m} \qquad \text{(eq. 3.72)}$$

$$\Rightarrow \quad A = \pi r^2 = \frac{l}{2m}T \quad \text{for uniform motion}$$

$$\Rightarrow \quad T = \frac{2m\pi r_0^2}{l} = \frac{2m\pi}{l}\left(\frac{-l^2}{a(n+1)m}\right)^{2/n+3}$$

b) We need to expand V_{eff} about r_0 to study a nearly circular orbit:

$$V_{eff}(r) \approx V_{eff}(r_0) + \frac{1}{2!}V_{eff}''(r_0)(r-r_0)^2$$

$$= -\frac{1}{2}ar_0^{n+1} + \frac{1}{2}\left(-a(n+1)\,n\,r_0^{n-1} + \frac{3l^2}{mr_0^4}\right)(r-r_0)^2$$

$$= \text{const} + \frac{1}{2r_0^4}\left(-a(n+1)n\left(\frac{-l^2}{a(n+1)m}\right) + \frac{3l^2}{m}\right)s^2$$

$$= \frac{l^2}{2r_0^4 m}(3+n)\,s^2 \equiv \frac{1}{2}ks^2$$

So the oscillation frequency is

$$\omega_{osc} = \sqrt{\frac{k}{m}} = \frac{\ell}{mr_0^2}\sqrt{n+3}$$

Compare with

$$\omega_{rot} = \frac{2\pi}{T} = \frac{\ell}{mr_0^2}$$

Orbits will close iff $\sqrt{n+3}$ is rational.

Also need $n < -1$ in order for orbits to exist at all.

Only integers that work are $n = -2, -3$.

3. First minimize $V_{eff}(r) = -\frac{k}{r} - \frac{\epsilon}{r^3} + \frac{\ell^2}{2mr^2}$:

$$V'_{eff}(r) = \frac{k}{r^2} + \frac{3\epsilon}{r^4} - \frac{\ell^2}{mr^3} = 0$$

$$\Rightarrow \quad kr^2 - \frac{\ell^2}{m}r + 3\epsilon = 0$$

$$r_0 = \frac{1}{2k}\left(\frac{\ell^2}{m} \pm \sqrt{\left(\frac{\ell^2}{m}\right)^2 - 12k\epsilon}\right)$$

$$\approx \frac{1}{2k}\left(\frac{\ell^2}{m} + \frac{\ell^2}{m}\left(1 - 6k\epsilon\left(\frac{m}{\ell^2}\right)^2\right)\right)$$

$$= \frac{\ell^2}{mk} - \frac{3\epsilon m}{\ell^2} = \frac{\ell^2}{mk}\left(1 - \frac{3\epsilon m^2 k}{\ell^4}\right)$$

$$V''_{eff}(r_0) = -\frac{2k}{r_0^3} - \frac{12\epsilon}{r_0^5} + \frac{3\ell^2}{mr_0^4}$$

$$\approx k\left(\frac{mk}{\ell^2}\right)^3 + 6\epsilon\left(\frac{mk}{\ell^2}\right)^5 + O(\epsilon^2)$$

Then the effective Lagrangian for r is

$$L = \tfrac{1}{2} m \dot{r}^2 - V_{eff}(r) \approx \tfrac{1}{2} m \dot{r}^2 - V_{eff}''(r_0)(r-r_0)^2$$

so the oscillation frequency is

$$\omega_{osc}^2 = \frac{V_{eff}''(r_0)}{m} = \frac{k}{m}\left(\frac{mk}{l^2}\right)^3 + \frac{6\varepsilon}{m}\left(\frac{mk}{l^2}\right)^5$$

or

$$\omega_{osc} = \sqrt{\frac{m^2k^4}{l^6}\left(1+6\varepsilon\frac{m^2k}{l^4}\right)} \approx \frac{mk^2}{l^3}\left(1+3\varepsilon\frac{m^2k}{l^4}\right)$$

[using $\sqrt{1+x} \approx 1+2x$ for small x.]

Compare this to

$$\omega_{rot} = \frac{2\pi}{T} = \frac{l}{mr_0^2} = \frac{l}{m\cdot\left[\frac{l^2}{mk}\left(1-\frac{3\varepsilon m^2k}{l^4}\right)\right]^2}$$

$$\approx \frac{mk^2}{l^3}\left(1+6\frac{k\varepsilon m^2}{l^4}\right)$$

The fractional difference in periods

$$\frac{\Delta T}{T} = \frac{\frac{2\pi}{\omega_{osc}} - \frac{2\pi}{\omega_{rot}}}{\frac{2\pi}{\omega_{rot}}} \approx 3\varepsilon\frac{km^2}{l^4} = 3\left(\frac{kl^2}{mc^2}\right)\frac{km^2}{l^4} = 3\frac{k^2}{c^2l^2}.$$

This should also equal $\frac{\Delta\theta}{2\pi}$ for near-uniform motion, so the angular precession per period is

$$\frac{\Delta\theta}{T} = 2\pi\frac{\Delta T}{T^2} = 6\pi\frac{k^2}{c^2l^2T}$$ in agreement with eq. (12.49).

4. A scattering orbit consists of 3 parts:

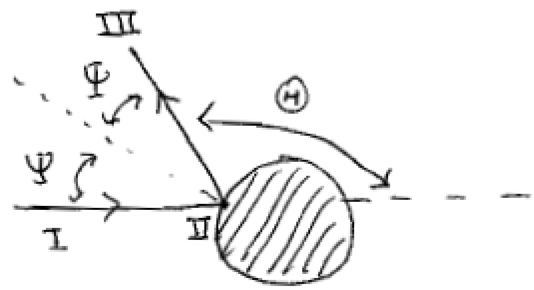
I. Free particle propagation parallel to the x-axis from ∞ to $r = a$.

II. Instantaneous scattering interaction with the sphere.

III. Free particle propagation back out to $r = \infty$, in a direction making an angle Θ with $+x$-axis.

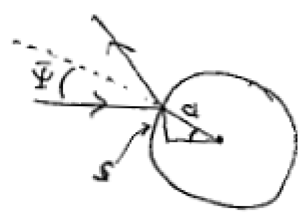
Eq. (3.94), valid for any spherically symmetric potential, relates the scattering angle Θ to the angle between the incoming segment I and the radial line through the periapsis

$$\Theta = \pi - 2\Psi$$



The impact parameter and Ψ are related by

$$\sin\Psi = \frac{s}{a}$$



Hence

$$\Theta(s) = \pi - 2\sin^{-1}\left(\frac{s}{a}\right)$$

That's for $s \le a$. For $s > a$, the particle doesn't scatter and $\Theta = 0$.

(b) Inverting the above relation for $\Theta(s)$ gives

$$s(\Theta) = a \sin\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = a \cos\left(\frac{\Theta}{2}\right).$$

Therefore

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \frac{a\cos\left(\frac{\Theta}{2}\right)}{\sin\Theta} \cdot \frac{a}{2}\sin\frac{\Theta}{2} = \frac{a^2}{4}$$

The total cross section is

$$\sigma_{tot} = \int \sigma(\Theta)\, d\Omega = \int \frac{a^2}{4}\, d\Omega = \frac{a^2}{4} \cdot 4\pi = \pi a^2$$

This is precisely the cross-sectional area of the sphere, a reflection of the fact that everything that hits the sphere gets scattered, and no orbits with $s > a$ get deflected at all.