

1. (a) Referring to Problem 5.2:

Only orbits of type (i) are scattering orbits.

The rest get sucked to $r=0$.

Following the discussion on p. 109, take the periapsis to lie on $\theta=0$.

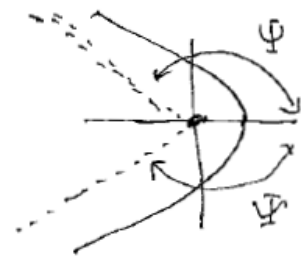
Then setting $r=\infty$ in the orbit equation gives Ψ :

$$0 = \frac{1}{r} = \cos \sqrt{\frac{\lambda^2 - 2mk}{\lambda^2}} \Psi \quad \Rightarrow \quad \Psi = \pm \frac{\pi}{2} \sqrt{\frac{\lambda^2}{\lambda^2 - 2mk}}$$

(There are other solutions, corresponding to orbits that circle $\bar{r}=0$ multiple times - we ignore these for simplicity.)

Note this $|\Psi|$ is bigger than $\frac{\pi}{2}$, so the relation between Ψ and the scattering angle Θ is different (by 2π):

$$\begin{aligned} \Theta &= -\pi + 2\Psi \\ &= -\pi + \pi \sqrt{\frac{\lambda^2}{\lambda^2 - 2mk}} \\ &= \pi \left(-1 + \sqrt{\frac{s^2 E}{s^2 E - k}} \right) \end{aligned}$$



where in the last line we used $\lambda = s\sqrt{2mE}$.

Now invert to get

$$s^2 = \frac{k}{E} \frac{(\Theta + \pi)^2}{\Theta(\Theta + 2\pi)} \quad \frac{ds^2}{d\Theta} = 2s \left| \frac{ds}{d\Theta} \right| = \frac{2k}{E} \frac{\pi^2 (\Theta + \pi)}{\Theta^2 (\Theta + 2\pi)^2}$$

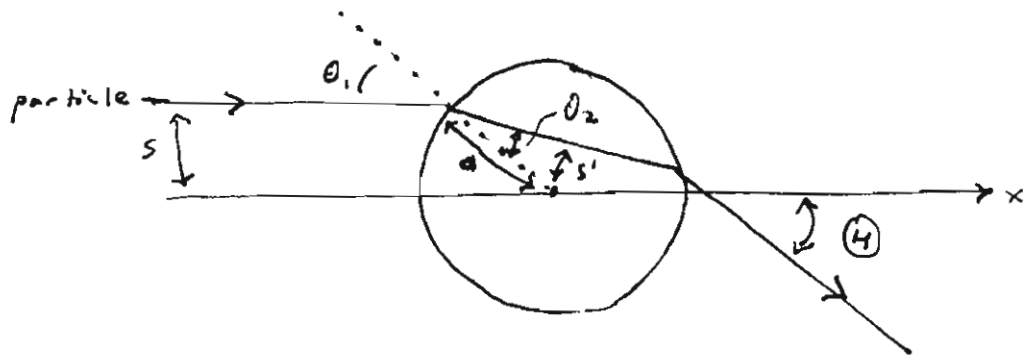
and

$$\sigma(\Theta) = \frac{k \pi^2}{E} \frac{\Theta + \pi}{\sin \Theta \cdot \Theta^2 (\Theta + 2\pi)^2}$$

(b) For given $E > 0$ (required for a scattering orbit) the orbits which get captured rather than scattered are of type (iii), with $l^2 < 2mk$:
i.e. $s^2 < \frac{k}{E}$ (using $s\sqrt{2mE} = l$). Thus the cross-sectional area of the beam that gets captured is

$$\sigma_{\text{capt}} = \pi S_{\text{max}}^2 = \pi \frac{k^2}{E^2}.$$

2. The particle's scattering orbit will be as shown:



For $r > a$, the particle will follow a straight-line path of velocity $v = \sqrt{2mE}$ (since $E = \frac{1}{2}mv^2$ if $r > a$).

For $r < a$, the path will also be a straight line, with velocity $v' = \sqrt{2m(E+V_0)}$. Hence $v' = \sqrt{\frac{E+V_0}{E}} v$.

To determine the complete path, we only need to find the angles of "refraction" at the boundary of the sphere.

Conservation of angular momentum \Rightarrow

$$\begin{aligned}
 mvs &= mv's' \\
 \Rightarrow mvs &= mv \sqrt{\frac{E+V_0}{E}} s' \\
 \Rightarrow s &= \sqrt{\frac{E+V_0}{E}} s'
 \end{aligned}$$

From the figure, $\sin \theta_1 = \frac{s}{a}$ and $\sin \theta_2 = \frac{s'}{a}$.

So $\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{\frac{E+V_0}{E}}$ which is exactly what Snell's

Law gives for light refracted by a medium with index of refraction $n = \sqrt{\frac{E+V_0}{E}}$. Thus the path of the particle is the same as the path of light

scattered by a refracting sphere.

To find $\sigma(\theta)$, we need to determine θ as a function of s . Again referring to the figure,

$$\begin{aligned}\frac{\theta}{2} &= \theta_1 - \theta_2 = \sin^{-1}\left(\frac{s}{a}\right) - \sin^{-1}\left(\frac{s'}{a}\right) \\ &= \sin^{-1}\left(\frac{s}{a}\right) - \sin^{-1}\left(\frac{s}{a} \sqrt{\frac{E}{E+V_0}}\right)\end{aligned}$$

Taking the cosine of both sides and using trig identities gives

$$s(\theta) = \frac{na \sin\left(\frac{\theta}{2}\right)}{\sqrt{n^2 + 1 - 2n \cos\left(\frac{\theta}{2}\right)}} \quad \text{where } n \equiv \sqrt{\frac{E+V_0}{E}}$$

Now plug into eq. (3.93) to get the desired result.

$$3.(a) \quad L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} e B r^2 \dot{\theta} + \frac{1}{2} m \omega^2 r^2$$

$$(b) \quad E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = m \dot{r}^2 + (m r^2 \dot{\theta} + \frac{1}{2} e B r^2) \dot{\theta} - L$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} m \omega^2 r^2$$

$$l = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} + \frac{1}{2} e B r^2$$

(c) Substituting $\dot{\theta} = \frac{l}{m r^2} - \frac{e B}{2m}$ into E gives

$$E = \frac{1}{2} m \left[\dot{r}^2 + r^2 \left(\frac{l}{m r^2} - \frac{e B}{2m} \right)^2 \right] - \frac{1}{2} m \omega^2 r^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2m r^2} + \underbrace{\frac{1}{2} m \left[\left(\frac{e B}{2m} \right)^2 - \omega^2 \right] r^2}_{V_{\text{eff}}(r)} + \text{constant}$$

The effective potential is identical to that of a 2D harmonic potential of frequency ω_0

$$\omega_0^2 = \left(\frac{e B}{2m} \right)^2 - \omega^2$$

The orbits are bounded if and only if $\omega_0^2 \geq 0$;

$$\left(\frac{e B}{2m} \right)^2 \geq \omega^2$$

Thus, a magnetic field of sufficient strength can stabilize an inverted oscillator.

5. (a) Proved the same as for real numbers.

(b) Ditto.

$$\begin{aligned} (c) \quad e^{CBC^{-1}} &= \mathbb{1} + CBC^{-1} + \frac{1}{2}(CBC^{-1})^2 + \dots \\ &= C\mathbb{1}C^{-1} + CBC^{-1} + \frac{1}{2}CB^2C^{-1} + \dots \\ &= C(\mathbb{1} + B + \frac{1}{2}B^2 + \dots)C^{-1} \end{aligned}$$

$$(d) \quad \tilde{A} = e^{\tilde{B}} = e^{\tilde{B}} = e^{-B} = A^{-1}.$$

$$(e) \quad \exp \begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \\ -\theta & 0 & 0 \end{pmatrix} = \cos \theta \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin \theta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$