

PHY 504 Problem Set #6 Solutions

1. (a+b) See solution to problem 4(e), set #4.

(c) Conserved quantities are:

$$(i) p_x = m\dot{x} - qBy$$

$$(ii) p_y = m\dot{y} + qBx$$

$$(iii) p_z = m\dot{z}$$

$$(iv) L = m(\dot{x}y - \dot{y}x) - \frac{1}{2}qB(x^2 + y^2)$$

$$(v) E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Take the time derivative of (i)

$$0 = m\ddot{x} - qB\dot{y}$$

and plug in $\dot{y} = \frac{p_y - qBx}{m}$ from (ii) to get

$$\ddot{x} = -\left(\frac{qB}{m}\right)^2 \left(x - \frac{p_y}{qB}\right)$$

Similarly

$$\ddot{y} = -\left(\frac{qB}{m}\right)^2 \left(y + \frac{p_x}{qB}\right)$$

This is an oscillator of frequency $\omega = \frac{qB}{m}$ centered at $y_0 = -\frac{p_x}{qB}$. Likewise x oscillates

about $x_0 = \frac{p_y}{qB}$. Thus

$$x = x_0 + A_1 \cos(\omega t + \phi_1)$$

$$y = y_0 + A_2 \cos(\omega t + \phi_2)$$

and

$$\dot{x} = -A_1 \omega \sin(\omega t + \phi_1)$$

$$\dot{y} = -A_2 \omega \sin(\omega t + \phi_2)$$

Conservation of P_z and E , (iii) and (iv), imply

that $\dot{x}^2 + \dot{y}^2$ is a constant:

$$\dot{x}^2 + \dot{y}^2 = \omega^2 (A_1^2 \sin^2(\omega t + \varphi_1) + A_2^2 \sin^2(\omega t + \varphi_2)) = \text{constant}$$

Shifting t so that $\varphi_1 = 0$, the only solutions

are $A_1 = \pm A_2$ and $\varphi_2 = \pm \frac{\pi}{2}$, and

$$x = x_0 + A \cos \omega t$$

$$y = y_0 \pm A \sin \omega t$$

This general solution* describes circular motion of radius A about (x_0, y_0) , in a counterclockwise or clockwise direction depending on the \pm sign.

In fact, in order to conserve l the '-' sign must be chosen.

For circular motion about the origin in the (xy) -plane,

$$E = \frac{1}{2} m \omega^2 A^2 \quad \text{and} \quad l = 0$$

$$= \frac{1}{2} \frac{q^2 B^2}{m} A^2$$

$$\text{or} \quad A = \frac{\sqrt{2mE}}{qB}$$

* ignoring the trivial motion in the z -direction.

2. Only orbits of type (i) are scattering orbits.

The rest get sucked to $r=0$.

Following the discussion on p. 109, take the periastris to lie on $\theta=0$.

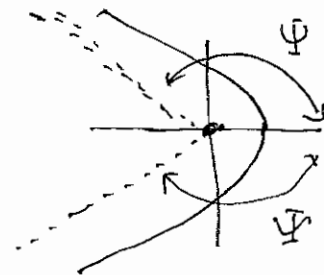
Then setting $r=\infty$ in the orbit equation gives Ψ :

$$0 = \frac{1}{r} = \cos \sqrt{\frac{l^2 - 2mk}{l^2}} \Psi \quad \Rightarrow \quad \Psi = \pm \frac{\pi}{2} \sqrt{\frac{l^2}{l^2 - 2mk}}$$

(There are other solutions, corresponding to orbits that circle $\bar{r}=0$ multiple times - we ignore these for simplicity.)

Note this $|\Psi|$ is bigger than $\frac{\pi}{2}$, so the relation between Ψ and the scattering angle Θ is different (by 2π):

$$\begin{aligned} \Theta &= -\pi + 2\Psi \\ &= -\pi + \pi \sqrt{\frac{l^2}{l^2 - 2mk}} \\ &= \pi \left(-1 + \sqrt{\frac{s^2 E}{s^2 E - k}} \right) \end{aligned}$$



where in the last line we used $l = s\sqrt{2mE}$.

Now invert to get

$$s^2 = \frac{k}{E} \frac{(\Theta + \pi)^2}{\Theta(\Theta + 2\pi)} \quad \frac{ds^2}{d\Theta} = 2s \left| \frac{ds}{d\Theta} \right| = \frac{2k}{E} \frac{\pi^2 (\Theta + \pi)}{\Theta^2 (\Theta + 2\pi)^2}$$

and

$$\sigma(\Theta) = \frac{k \pi^2}{E} \frac{(\Theta + \pi)}{\sin^2 \Theta \cdot (\Theta^2 (\Theta + 2\pi)^2)}$$

b For given $E_z > 0$ (required for a scattering orbit)
the orbits which get captured rather than
scattered are of type (iii), with $l^2 < 2mk$:

i.e. $s^2 < \frac{k}{E}$ (using $s\sqrt{2mE} = l$). Thus the
cross-sectional area of the beam that gets
captured is

$$\sigma_{\text{capt}} = \pi S_{\text{max}}^2 = \pi \frac{k^2}{E^2}.$$

3. The sign of the scattering angle in (3.101) would change but not its magnitude; the cross section in (3.102) would be unchanged since it only depends on the magnitudes of Z and Z' and not on their relative sign.

4. $\widetilde{AB} = \widetilde{B}\widetilde{A}$:

In components,

$$(\widetilde{AB})_{ij} = (AB)_{ji} = A_{jk} B_{ki} = \widetilde{A}_{kj} \widetilde{B}_{ik} = \widetilde{B}_{ik} \widetilde{A}_{kj} = (\widetilde{B}\widetilde{A})_{ij}$$

$(AB)^+ = B^+ A^+$: Proved exactly the same way.

Finally, if $A^{-1} = \widetilde{A}$ and $B^{-1} = \widetilde{B}$, then

$$(\widetilde{AB}) = \widetilde{B}\widetilde{A} = B^{-1}A^{-1} = (AB)^{-1} \quad \checkmark$$

5. (a) Proved the same as for real numbers.

(b) Ditto.

$$\begin{aligned} \text{(c)} \quad e^{CBC^{-1}} &= \mathbb{I} + CBC^{-1} + \frac{1}{2}(CBC^{-1})^2 + \dots \\ &= C\mathbb{I}C^{-1} + CBC^{-1} + \frac{1}{2}CB^2C^{-1} + \dots \\ &= C(\mathbb{I} + B + \frac{1}{2}B^2 + \dots)C^{-1} \\ &= Ce^B C^{-1} = e^{AC^{-1}}. \end{aligned}$$

$$\text{(d)} \quad \widetilde{A} = e^{\widetilde{B}} = e^{\widetilde{B}^{-1}} = e^{-B} = A^{-1}.$$

$$\text{(e)} \quad \exp \begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \\ -\theta & 0 & 0 \end{pmatrix} = \cos \theta \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin \theta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$