PHY 504 Problem Set #7 Solutions

1.(a) $\epsilon_{ijp}\epsilon_{rmp} = \delta_{ir}\delta_{jm} - \delta_{im}\delta_{jr}$:

If $i=j$ or $r=m$, both sides vanish.

Otherwise, $i\neq j$ and $r\neq m$.

If $i\neq j$ then suppose $i=1$ and $j=2$ without loss of generality.

Then only the term with $p=3$ will be nonzero in the sum on RHS.

If r or $m=3$, both sides vanish.

Otherwise r and $m\neq 3$, so either $r=1$ and $m=2$ or $r=2, m=1$.

If $r=1$ and $m=2$, then

$$\epsilon_{123}\epsilon_{123} = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} = 1$$

If $r=2$ and $m=1$, then

$$\epsilon_{123}\epsilon_{213} = \delta_{12}\delta_{21} - \delta_{11}\delta_{22} = -1 .$$

For all other cases, the identity follows by permuting the labels $\{1,2,3\}$.

(b) $\epsilon_{ijp}\epsilon_{ijk} = \epsilon_{pij}\epsilon_{kij} = \sum_i \left(\delta_{pk}\delta_{ii} - \delta_{pi}\delta_{ki}\right)$

$$= \delta_{pk}\sum_i \delta_{ii} - \delta_{pk} = 2\delta_{pk} .$$

(c) $\left[(\vec{a}\times\vec{b})\times\vec{c}\right]_i = \epsilon_{ijk}(a\times b)_j c_k$

$$= \epsilon_{ijk}\epsilon_{jlm}a_l b_m c_k$$

$$= -\epsilon_{ikj}\epsilon_{lmj}a_l b_m c_k$$

$$= -(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl})a_l b_m c_k$$

$$= -a_i b_k c_k + a_k b_i c_k$$

$$= \left[-\vec{a}(\vec{b}\cdot\vec{c}) + \vec{b}(\vec{a}\cdot\vec{c})\right]_i$$

(d) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$.

So cross product is not associative.

(e) $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = \epsilon_{ijk}a_j b_k \epsilon_{ilm}a_l b_m$

$$= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})a_j b_k a_l b_m$$

$$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$$

$$= |a|^2|b|^2 - (\vec{a}\cdot\vec{b})^2$$

2. (a) As discussed in class, the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

implements a rotation by acting on column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$.

Similarly,

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\, x + \sin\theta\, y \\ -\sin\theta\, x + \cos\theta\, y \\ 1 \end{pmatrix}$$

rotates (x, y) by angle θ.

To produce a translation by (a_x, a_y) we use

$$\begin{pmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + a_x \\ y + a_y \\ 1 \end{pmatrix}$$

A rotation followed by a translation is implemented by the product of these 2 matrices:

$$\begin{pmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & a_x \\ -\sin\theta & \cos\theta & a_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} (\cos\theta\, x + \sin\theta\, y) + a_x \\ (-\sin\theta\, x + \cos\theta\, y) + a_y \\ 1 \end{pmatrix}$$

$\uparrow$

So this matrix represents a rotation followed by a translation.

(b) We must verify:

- closure: the product of 2 matrices of this form is another matrix of the same form. Straightforward, but we'll omit it here.
- associativity: true for matrix multiplication in general.
- inverse:

$$\begin{pmatrix} \cos\theta & \sin\theta & a_x \\ -\sin\theta & \cos\theta & a_y \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -a_x \\ 0 & 1 & -a_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & -\cos\theta\, a_x + \sin\theta\, a_y \\ \sin\theta & \cos\theta & -\cos\theta\, a_y - \sin\theta\, a_x \\ 0 & 0 & 1 \end{pmatrix}$$

- identity: $\theta = 0^\circ$ and $a_x = a_y = 0$ gives $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

<u>Nonabelian</u>: It's easily checked that performing a translation first, followed by a rotation, does not give the same result as the rotation followed by the same translation.

The closure property implies that any sequence of translations and rotations, that is, any element of the Euclidean group, can be represented by a matrix of this form.

3. (a) Rotation of 60° about x-axis:

$$R\left(\frac{\pi}{3}\hat{x}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Rotation of 45° about z-axis:

$$R\left(\frac{\pi}{4}\hat{z}\right) = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Net rotation

$$\underbrace{R\left(\frac{\pi}{4}\hat{z}\right)}_{\uparrow\ 2^{nd}\text{ rotation}} \underbrace{R\left(\frac{\pi}{3}\hat{x}\right)}_{\uparrow\ 1^{st}\text{ rotation}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Reversing the order:

$$R\left(\frac{\pi}{3}\hat{x}\right) R\left(\frac{\pi}{4}\hat{z}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

4. Note by inspection that the (jk)-component of $\underset{\sim}{M}_i$ is:

$$(\underset{\sim}{M}_i)_{jk} = -\epsilon_{ijk}$$

So $$[M_i, M_j]_{lm} = (M_i)_{lp}(M_j)_{pm} - (M_j)_{lp}(M_i)_{pm}$$

$$= \epsilon_{ilp}\epsilon_{jpm} - \epsilon_{jlp}\epsilon_{ipm}$$

$$= -\epsilon_{ilp}\epsilon_{jmp} + \epsilon_{jlp}\epsilon_{imp}$$

$$= -\cancel{\delta_{ij}\delta_{lm}} + \delta_{im}\delta_{lj} + \cancel{\delta_{ji}\delta_{lm}} - \delta_{jm}\delta_{li}$$

$$= \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}$$

$$= \epsilon_{ijk}\epsilon_{mlk} = \epsilon_{ijk}(-\epsilon_{klm})$$

$$= \epsilon_{ijk}(M_k)_{lm}$$