PHY 504 Problem Set #8 Solutions

1. In the rotating frame,

$$\ddot{\vec{r}} = -2(\vec{\omega}\times\dot{\vec{r}}) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) + m\vec{g}$$

The Coriolis and centrifugal accelerations are both in the horizontal xy-plane.

The horizontal components of the equation of motion are

$$\begin{cases} \ddot{x} = 2\omega\dot{y} + \omega^2 x \\ \ddot{y} = -2\omega\dot{x} + \omega^2 x \end{cases}$$

exactly as found in problem 1 of PS#1.

These may be solved by various techniques; however we have already found the solution, so let's use it:

$$\begin{cases} x(t) = (x_0 - \omega y_0 t)\cos\omega t + (y_0 + \omega x_0 t)\sin\omega t \\ y(t) = -(x_0 - \omega y_0 t)\sin\omega t + (y_0 + \omega x_0 t)\cos\omega t \end{cases}$$

Let's take $x_0 = r_0$, $y_0 = 0$.

If the ball is dropped from height $h = \frac{1}{2}gt^2$,

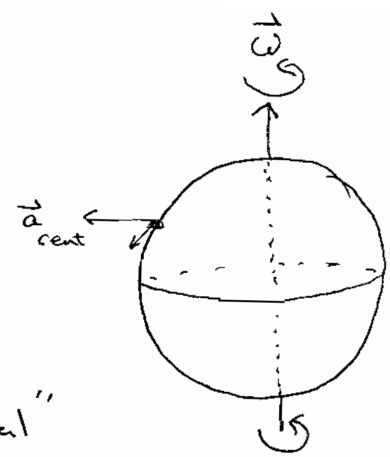
then $t_f = \sqrt{\frac{2h}{g}}$ and

$$x(t_f) = r_0 \cos\omega\sqrt{\frac{2h}{g}} + \omega r_0\sqrt{\frac{2h}{g}}\sin\omega\sqrt{\frac{2h}{g}}$$

$$y(t_f) = -r_0 \sin\omega\sqrt{\frac{2h}{g}} + \omega r_0\sqrt{\frac{2h}{g}}\cos\omega\sqrt{\frac{2h}{g}}$$

are the final coordinates of the ball.

2. (a) The centrifugal acceleration points radially away from the N-S axis of Earth and at 38° N latitude has a "horizontal" component pointing to the south, of magnitude



$$a_{cent,s} = -\left[\vec{\omega} \times (\vec{\omega} \times \vec{r})\right]_S$$

$$= \left[\omega^2 R \quad \cos 38° \, \hat{r}\right]_S$$

$$= \omega^2 R \cos 38° \sin 38°$$

$$= 0.0338 \, m/s^2 \cdot 0.485 = 0.0164 \, m/s^2$$

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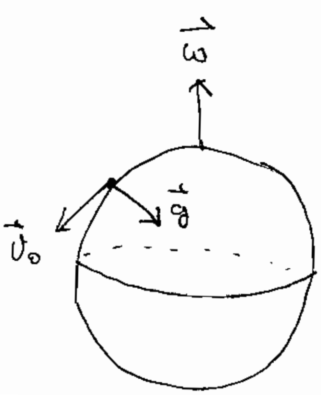
This leads to no horizontal deflection for the bullet fired South.

During the 10 s the bullet is in the air, the effect of gravity on its velocity is significant, and must be taken into account:

$$\vec{v}(t) = \vec{v}_0 + \vec{g}t \qquad \text{where } \vec{g} = (-9.8 \, m/s^2)\,\hat{r}$$



Then

$$\vec{a}_{cor} = -2(\vec{\omega} \times \vec{v}) = -2\,\vec{\omega} \times (\vec{v}_0 + \vec{g}t)$$

$$= -2\left(\omega v_0 \sin 38° \, \hat{E} - \omega g t \cos 38° \, \hat{E}\right)$$

unit vector to East

So the Coriolis acceleration leads to a net deflection to the East of

$$\Delta X_{cor} = -2\left(\omega U_0 \sin 38^\circ \frac{t^2}{2} - \omega g \frac{t^3}{3} \cos 38^\circ\right)$$

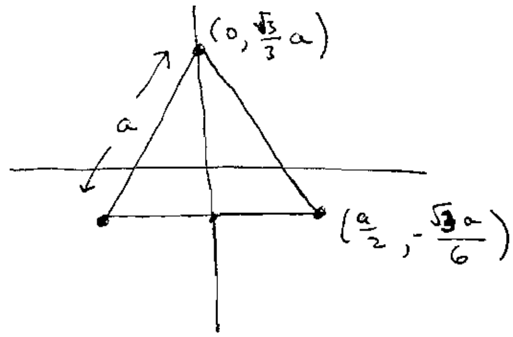
$$= -2\left(7.3\times10^{-5}\,s^{-1}\right)\left[(100\,m/s)(0.6)\frac{10^2}{2} - 9.8\,m/s^2\,(0.8)\frac{(10\,s)^3}{3}\right]$$

$$= -0.056\,m$$

The bullet will be deflected to the west by 0.056 m.

(b) Left to the reader.

3. We will calculate $\underline{\underline{I}}$ about the center of mass



$$I_{xx} = \rho \int_{-a/2}^{a/2} dx \int_{-\frac{\sqrt{3}a}{6}}^{\frac{\sqrt{3}}{3}a - \sqrt{3}|x|} dy \left((x^2+y^2) - x^2 \right) \qquad \text{where } \rho = \frac{m}{\text{area}} = \text{surface density}$$

$$= 2 \cdot \left(\frac{4m}{\sqrt{3}\, a^2} \right) \int_0^{a/2} dx \int_{-\frac{\sqrt{3}a}{6}}^{\frac{\sqrt{3}}{3}a - \sqrt{3}x} dy \cdot y^2$$

$$= \frac{8m}{\sqrt{3}\, a^2} \cdot \int_0^{a/2} dx \left[\frac{1}{3}\left(\frac{\sqrt{3}}{3}a - \sqrt{3}x \right)^3 - \frac{1}{3}\left(-\frac{\sqrt{3}a}{6} \right)^3 \right]$$

$$= \frac{8m}{\sqrt{3}\, a^2} \left[\frac{1}{12}\left(\frac{\sqrt{3}}{3}a - \sqrt{3}x \right)^4 \cdot \left(-\frac{1}{\sqrt{3}} \right) + \frac{1}{3}\left(\frac{\sqrt{3}a}{6} \right)^3 \cdot x \right]_0^{a/2}$$

$$= \frac{8m}{\sqrt{3}\, a^2} \left[\frac{1}{12}\left(\frac{\sqrt{3}}{6}a \right)^4 \left(-\frac{1}{\sqrt{3}} \right) - \frac{1}{12}\left(\frac{\sqrt{3}}{3}a \right)^4 \left(-\frac{1}{\sqrt{3}} \right) + \frac{1}{3}\left(\frac{\sqrt{3}a}{6} \right)^3 \cdot \frac{a}{2} \right]$$

$$= \frac{8m}{\sqrt{3}\, a^2} \left(\frac{\sqrt{3}}{6}a \right)^4 \left[\frac{1}{12}\left(-\frac{1}{\sqrt{3}} \right) - \frac{16}{12}\left(-\frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{3}} \right]$$

$$= \frac{8m}{3a^2} \cdot \frac{9}{6^4} a^4 \cdot \frac{27}{12} = \frac{1}{24} m a^2$$

4.(a) With the edges of the cube parallel to the x-, y-, and z-axes and a corner at the origin, the diagonal components of $\underline{I}$ are equal to

$$I_{xx} = \sum_n m_\cdot \left(r_{(n)}^2 - x_{(n)}^2 \right)$$

$$= m \cdot \left[3a^2 + 3(\sqrt{2}\,a)^2 + (\sqrt{3}\,a)^2 - 4a^2 \right]$$

$$= 8ma^2$$

Symmetry also implies that the off-diagonal components are equal to

$$I_{xy} = \sum_n m\left(-x_{(n)} y_{(n)}\right)$$

$$= -2ma^2$$

So

$$\underline{I} = ma^2 \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

The long diagonal of the cube is the vector $\vec{d} = (1, 1, 1)$. Check if this is eigenvector:

$$\underline{I}\,\vec{d} = ma^2 \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 4ma^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

It is an eigenvector with eigenvalue $4ma^2$.

(b) Choosing coordinates so that the z-axis is aligned with the long axis of the cube, $\hat{d} = -\hat{z}$, $\hat{z}$ will now be an eigenvector of $\underset{\sim}{I}$, so that

$$\underline{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & 4ma^2 \end{pmatrix}$$

eigenvalue of I in $\hat{z}$ direction

$\hat{z}$ is a principal axis with moment of inertia $I_3 = 4ma^2$.

Since $\vec{L}_3 = I\vec{\omega}$ is conserved in the space frame$_3$ in the absence of torques,

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(I\vec{\omega}) = \frac{d}{dt}(I_3\omega_3) = I_3\dot{\omega}_3 = 0$$

if the cube rotates about the z-axis, so the cube will spin with constant ω_3.

If a torque N_3 is applied in the z-direction, then $I_3\dot{\omega}_3 = N_3$ and the angular acceleration is

$$\alpha = \dot{\omega}_3 = \frac{N_3}{I_3} = \frac{N_3}{4ma^2}.$$

Similarly

$$I_{yy} = 2\rho \int_0^{a/2} dx \int_{-\frac{\sqrt{3}}{6}a}^{\frac{\sqrt{3}}{2}a - \sqrt{3}x} dy \; x^2 = 2\rho \int_0^{a/2} dx \; x^2 \cdot \left(\frac{\sqrt{3}}{2}a - \sqrt{3}\,x\right)$$

$$= 2\rho \left(\frac{\sqrt{3}}{2} a \frac{x^3}{3} - \sqrt{3}\frac{x^4}{4} \right)\Bigg|_0^{a/2} = \frac{1}{24} m a^2 .$$

$$I_{xy} = \underset{\triangle}{\int dx \int dy} \; (-xy) = 0$$

by antisymmetry ~~about~~ of integrand under $x \to -x$.

$$I_{zz} = \iint dx\,dy\,(x^2+y^2) = I_{xx} + I_{yy} = \frac{1}{12} m a^2$$

$$I_{zx} = I_{zy} = 0 \quad \text{as for } I_{xy} .$$

Finally

$$I = \begin{pmatrix} \frac{1}{24} m a^2 & 0 & 0 \\ 0 & \frac{1}{24} m a^2 & 0 \\ 0 & 0 & \frac{1}{12} m a^2 \end{pmatrix}$$

(c) The eigenvalues of I, i.e. its principal moments, are obtained by solving

$$\det(I - \lambda \mathbb{1}) = 0$$

which gives (we omit the details) $\lambda = 4ma^2, 10ma^2, 10ma^2$.

We have already seen that $\lambda = 4ma^2$ is the moment for rotation about the vertical axis.

Thus $\lambda = 10ma^2$ must be the ~~eigenval~~ moments about the horizontal axes. (Since these moments are degenerate, they are unchanged under a horizontal rotation, so it doesn't matter how we choose our x- and y-axes.)

Let θ denote the angle made by the long diagonal of the cube to the vertical.

Then

$$L = \frac{1}{2} I_1 \dot{\theta}^2 + mg\underbrace{l}_{=\frac{\sqrt{3}}{2}a = \text{dist from pivot to center of mass}}\cos\theta$$

$$\approx \frac{1}{2} I_1 \dot{\theta}^2 - \frac{1}{2} mgl\,\theta^2$$

The frequency is $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mgl}{I_1}} = \sqrt{\frac{mg \cdot \frac{\sqrt{3}}{2}a}{10ma^2}}$

$$= \sqrt{\frac{\sqrt{3}}{20}\,\frac{g}{a}}.$$

5. (a) The cube rotates about the edge in contact with the table. The moment of inertia about this axis is

$$I = 4ma^2 + 2m(\sqrt{2}a)^2 = 8ma^2.$$

The gravitational potential energy is

$$V = 8mg\left(\frac{a}{\sqrt{2}}\right)\cos\theta$$

where $\frac{a}{\sqrt{2}}\cos\theta$ is the height of the center of mass above the table when the cube is tilting at an angle θ away from its initial position.

The conserved total energy of the cube is

$$E = \frac{1}{2} I \dot{\theta}^2 + \frac{8}{\sqrt{2}} mga\cos\theta.$$

Since the initial energy is $\frac{8}{\sqrt{2}}mga$, we have

$$\frac{1}{2} I \dot{\theta}^2 + \frac{8}{\sqrt{2}} mga\cos\theta = \frac{8}{\sqrt{2}} mga$$

or

$$\dot{\theta} = \sqrt{\frac{2}{I}(mga)\,\frac{8}{\sqrt{2}}\cdot(1-\cos\theta)}$$

$$= \sqrt{\frac{g}{a}(\sqrt{2}-1)}$$

when $\theta = 45^\circ$.

(b) Similar.