PHY 504 Problem Set #9 Solutions

1. For a rigid body in the absence of torques,

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2-I_3)$$
$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3-I_1)$$
$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1-I_2)$$

Assume that $I_1 > I_2 > I_3$.

First consider a perturbation to rotation about I_3.

ω_3 is approximately constant and ω_1 and ω_2 are small.

Then Euler's equation are approximated by

$$I_1\dot{\omega}_1 = \left[\omega_3(I_2-I_3)\right]\omega_2$$
$$I_2\dot{\omega}_2 = \left[\omega_3(I_3-I_1)\right]\omega_1$$
$$I_3\dot{\omega}_3 = 0$$

The first two equations combine to give

$$\ddot{\omega}_1 = \left[\omega_3^2\frac{(I_2-I_3)(I_3-I_1)}{I_1I_2}\right]\omega_1 \equiv -\Omega^2\omega_1$$

with a similar equation (same frequency) for ω_2.

The perturbed motion is stable because $\Omega^2 > 0$ if $I_1 > I_2 > I_3$.

The motion is a stable precession about the 3-axis, of frequency Ω.

Similarly, solutions close to a rotation about the 1-axis precess with frequency $\Omega^2 = \omega_1^2\frac{(I_1-I_2)(I_1-I_3)}{I_2I_3} > 0$.

But for motions close to the 2-axis,

$$\Omega^2 = \omega_2^2\frac{(I_1-I_2)(I_3-I_2)}{I_1I_3} < 0$$

Solutions for ω_1 and $\omega_3(t)$ will blow up exponentially, as $e^{|\Omega|t}$.

2. 2 conservation equations :

$$T = \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2$$

$$L^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2$$

Then
$$2TI_1 - L^2 = (I_1 I_2 - I_2^2)\,\omega_2^2 + (I_1 I_3 - I_3^2)\,\omega_3^2$$
$$2T I_2 - L^2 = (I_1 I_2 - I_1^2)\,\omega_1^2 + (I_2 I_3 - I_3^2)\,\omega_3^2$$

so
$$\omega_2 = \sqrt{\frac{2TI_1 - L^2 - (I_1 - I_3) I_3 \omega_3^2}{(I_1 - I_2) I_2 \cancel{2}}}$$
and ω_1 similarly.

The 3rd Euler equation gives

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\omega_1 \omega_2 =$$

$$= \frac{I_1 - I_2}{I_3}\left[\left(\frac{2TI_1 - L^2 - (I_1 - I_3) I_3 \omega_3^2}{(I_1 - I_2)\, I_2}\right)\left(\frac{2TI_2 - L^2 - (I_2 - I_3) I_3 \omega_3^2}{(I_2 - I_1)\, I_1}\right)\right]^{1/2}$$

$$= \frac{1}{I_3 \sqrt{I_2 I_1}}\left[-\left(2TI_1 - L^2 - (I_1 - I_3) I_3 \omega_3^2\right)\left(2TI_2 - L^2 - (I_2 - I_3) I_3 \omega_3^2\right)\right]^{1/2}$$

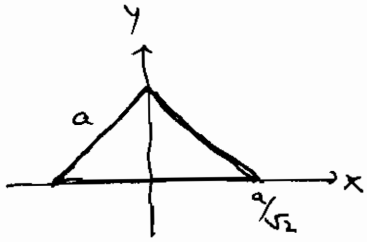
$$\sim A\left[B\omega_3^4 + C\omega_3^2 + D\right]^{1/2}$$

where A, B, C, D are constants depending on T, L^2, and I.
~~and $B > 0$~~. Then

$$t(\omega_3) = \int \frac{d\omega_3}{A\sqrt{B\omega_3^4 + C\omega_3^2 + D}}$$

This is an elliptic integral; inverting the elliptic function obtained will give $\omega_3(t)$.

3. First locate the center of mass:



Placing the triangle with its long side along the x-axis and its axis of symmetry along the y-axis, we calculate

$$\vec{R}_{CM} = \frac{\int d^2r\, \sigma(\vec{r})\, \vec{r}}{\int d^2r\, \sigma(\vec{r})} = \frac{\int d^2r\, \vec{r}}{(\text{Area})} = \frac{2}{a^2}\int d^2r\, \vec{r}$$

since the surface mass density σ is constant.

By symmetry $X_{cm} = 0$. And

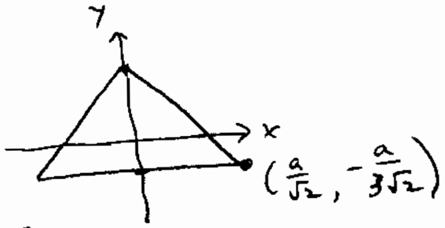
$$Y_{cm} = \frac{2}{a^2}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} dx \int_0^{a/\sqrt{2}-|x|} dy\; y$$

$$= \frac{2}{a^2}\cdot 2\int_0^{a/\sqrt{2}} dx \cdot \frac{1}{2}y^2\Big|_0^{a/\sqrt{2}-|x|}$$

$$= \frac{4}{a^2}\cdot\frac{1}{2}\cdot\frac{1}{3}\left(\frac{a}{\sqrt{2}}-x\right)^3(-1)\Big|_0^{a/\sqrt{2}} = \frac{a}{3\sqrt{2}}.$$

So the CM is at $\left(0, \frac{a}{3\sqrt{2}}\right)$.

Now shift to CM coords



and calculate the components of $\underline{\underline{I}}$.

(Note: With this orientation, we expect by symmetry $(x \to -x)$ that the y-axis will be a principal axis — that is, that the body will rotate about the y-axis w/o precessing. Similarly, by $z \to -z$ symmetry, we expect that the z-axis passing thru the CM will also be a principal axis. Since the 3 principal axes must be orthogonal, the x-axis is also principal.)

$$I_{yy} = 2\int_0^{\sqrt{a}/2} dx \int_{-\frac{a}{3\sqrt{2}}}^{2a/3\sqrt{2}-x} dy\,\sigma x^2 = 2\sigma \int_0^{\frac{\sqrt{a}}{2}} dx \cdot x^2\left(\frac{a}{\sqrt{2}} - x\right)$$

$$= 2\sigma\left(\frac{a\,x^3}{\sqrt{2}\cdot 3} - \frac{x^4}{4}\right)\Bigg|_0^{\sqrt{a}/2} = 2\left(\frac{2m}{a^2}\right)\left(\frac{a^4}{12} - \frac{a^4}{16}\right) = \frac{1}{12}ma^2$$

$$I_{xx} = 2\int_0^{\frac{a}{\sqrt{2}}} dx \int_{-\frac{a}{3\sqrt{2}}}^{\frac{2a}{3\sqrt{2}}-x} dy\;\sigma y^2 = 2\sigma\int_0^{\frac{\sqrt{a}}{2}} dx\;\frac{y^3}{3}\Bigg|_{-\frac{a}{3\sqrt{2}}}^{\frac{2a}{3\sqrt{2}}-x} = \dots = \frac{1}{36}ma^2$$

$$I_{zz} = \int dx \int dy\;\sigma(x^2+y^2) = I_{xx} + I_{yy} = \frac{1}{9}ma^2$$

$$I_{xy} = \int dx \int dy\;(-xy) = 0$$

since integrand is odd under $x \to -x$
and integration region is symmetric.

$$I_{xz} = \int dx \int dy \int dz\;\rho(\vec{r})\,(-xz) = 0$$

since $\rho(\vec{r}) = \delta(z)\cdot\sigma(x,y) = \sigma\cdot\delta(z)$.

$I_{yz} = 0$ similarly.

$$\underline{I} = ma^2\begin{pmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{9}\end{pmatrix}$$

The principal axes are the x, y, and z axes in our coordinates. This worked out nicely because we chose to orient the triangle in the most symmetric way possible.

4. For a top whose center of mass is fixed, the gravitational potential term in eq. (5.52) will disappear, leaving

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2$$

Solving the equations of motion for $u = \cos\theta(t)$ gives eq. (5.63) with $\beta = 0$:

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)\alpha - (b-au)^2}} = \int \frac{du}{\sqrt{\alpha - b^2 + 2abu - (a^2+\alpha)u^2}}$$

$$= \frac{1}{\sqrt{\alpha + a^2}} \sin^{-1} \frac{u - \frac{ab}{\alpha + a^2}}{\sqrt{\frac{\alpha - b^2}{\alpha + a^2} + \frac{1}{4}\left(\frac{2ab}{\alpha + a^2}\right)^2}} \Bigg|_{u(0)}^{u(t)}$$

$$= \frac{1}{\sqrt{\alpha + a^2}} \left\{ \sin^{-1} \frac{(\alpha + a^2)\, u(t) - ab}{\sqrt{\alpha^2 + (a^2 - b^2)\alpha}} - \varphi_0 \right\}$$

where $\varphi_0 = \sin^{-1} \frac{(\alpha + a^2)\, u(0) - ab}{\sqrt{\cdots}}$.

Inverting gives

$$u(t) = \cos\theta(t) = \frac{\sqrt{\alpha^2 + (a^2 - b^2)\alpha}\ \sin\left(\sqrt{\alpha + a^2}\, t + \varphi_0\right) + ab}{\alpha + a^2}$$

$$\equiv A \sin(\omega t + \varphi_0) + C$$

$\cos\theta$ oscillates about $\frac{ab}{\alpha + b^2}$ with frequency $\omega = \sqrt{\alpha + a^2}$.

$\phi(t)$ can be obtained as an elliptic integral or by numerical integration

$$\phi(t) = \int^t \frac{b - a\cos\theta}{\sin^2\theta}\, dt = \int^t \frac{b - au}{1 - u^2}\, dt$$