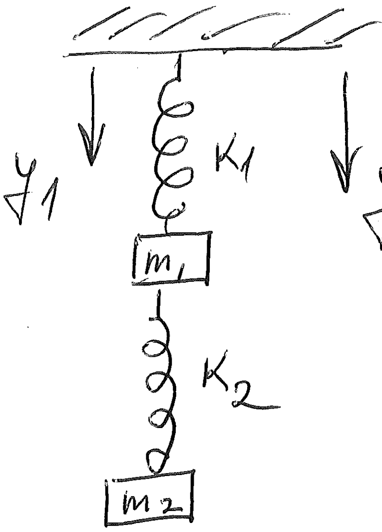
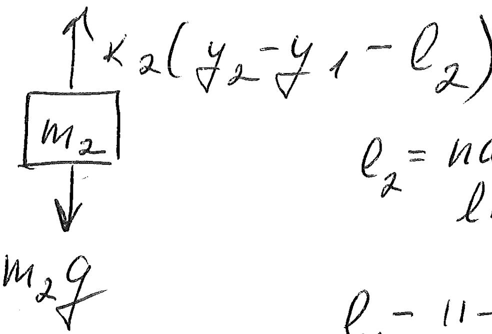
Set #11: Solutions

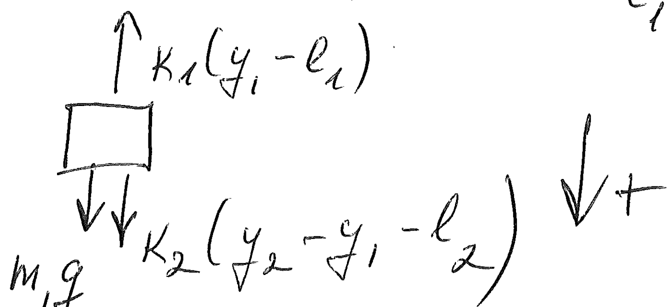
11.2





ℓ_2 = natural length of K_2

ℓ_1 — " — " — " of K_1



So,
$$m_1\ddot{y}_1 = m_1 g + K_2(y_2-y_1-\ell_2) - K_1(y_1-\ell_1)$$
$$m_2\ddot{y}_2 = m_2 g - K_2(y_2-y_1-\ell_2)$$

Set $\begin{cases} q_1 = y_1 - y_{1eq} \\ q_2 = y_2 - y_{2eq} \end{cases} \Rightarrow$
$$m_1\ddot{q}_1 = m_1 g + K_2(q_2-q_1+y_{2eq}-y_{1eq}-\ell_2) - K_1(q_1+y_{1eq}-\ell_1)$$
$$m_2\ddot{q}_2 = m_2 g - K_2(q_2-q_1+y_{2eq}-y_{1eq}-\ell_2)$$

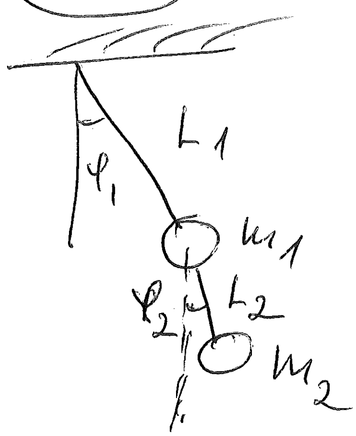
In equilibrium $q_1 = q_2 = \ddot{q}_1 = \ddot{q}_2 = 0$

$$\Rightarrow 0 = m_1 g + K_2(y_{2eq}-y_{1eq}-\ell_2) - K_1(y_{1eq}-\ell_1)$$
$$0 = m_2 g - K_2(y_{2eq}-y_{1eq}-\ell_2)$$
$$\Rightarrow m_1\ddot{q}_1 = K_2(q_2-q_1) - K_1 q_1$$
$$m_2\ddot{q}_2 = -K_2(q_2-q_1)$$

or $$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = -\begin{pmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

note, no g-dependence!

11.16



Eq. 11.43 gives the equations of motion

$$\underline{\underline{M}}\,\underline{\ddot{\varphi}} = -\underline{\underline{K}}\,\underline{\varphi}\ ; \quad \underline{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}$$

$$\underline{\underline{K}} = \begin{bmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

Setting $\underline{\varphi} = \underline{\varphi_0}\, e^{i\omega t}$ gives $\left[-\underline{\underline{M}}\omega^2+\underline{\underline{U}}\right]\underline{\varphi}=0$

or

$$\det\begin{bmatrix} (m_1+m_2)gL_1 - \omega^2(m_1+m_2)L_1^2 & -\omega^2 m_2 L_1 L_2 \\ -\omega^2 m_2 L_1 L_2 & m_2 g L_2 - \omega^2 m_2 L_2^2 \end{bmatrix} = 0$$

$$(m_1+m_2)L_1(g-\omega^2 L_1)\,m_2 L_2(g-\omega^2 L_2) - \omega^4 m_2^2 L_1^2 L_2^2 = 0$$

$$(m_1+m_2)m_2 L_1 L_2 g^2 - \omega^2\left[(m_1+m_2)L_1^2 m_2 L_2 g + (m_1+m_2)L_1 m_2 L_2^2 g\right] + \omega^4\left[(m_1+m_2)m_2 L_1^2 L_2^2 - m_2^2 L_1^2 L_2^2\right] = 0$$

$$\omega^4 L_1^2 L_2^2 m_1 m_2 - \omega^2 (m_1+m_2) m_2 L_1 L_2 g (L_1+L_2) + (m_1+m_2) m_2 L_1 L_2 g^2 = 0$$

$$\omega^4 L_1 L_2 m_1 m_2 - \omega^2 (m_1+m_2) m_2 g (L_1+L_2) + (m_1+m_2) m_2 g^2 = 0$$

$$\omega^4 - \omega^2 \frac{m_1+m_2}{m_1} g \frac{L_1+L_2}{L_1 L_2} + \frac{m_1+m_2}{L_1 L_2 m_1} g^2 = 0$$

$$\left[\omega^2 - \frac{m_1+m_2}{2m_1} g \frac{L_1+L_2}{L_1 L_2}\right]^2 - \frac{(m_1+m_2)^2}{4m_1^2} \frac{g^2 (L_1+L_2)^2}{L_1^2 L_2^2} + \frac{m_1+m_2}{m_1 L_1 L_2} g^2 = 0$$

$$\omega^2 = \frac{m_1+m_2}{2m_1} \frac{g(L_1+L_2)}{L_1 L_2} \pm \sqrt{\frac{(m_1+m_2)^2}{4m_1^2} g^2 \frac{(L_1+L_2)^2}{L_1^2 L_2^2} - \frac{m_1+m_2}{m_1 L_1 L_2} g^2}$$

$\Longrightarrow$

$m_1 = m_2 = m$

$L_1 = L_2 = L$

$$\frac{2g}{L} \pm \sqrt{\frac{4g^2}{L^2} - \frac{2g^2}{L^2}} = (2 \pm \sqrt{2}) \frac{g}{L} =$$

$$= (2 \pm \sqrt{2}) \omega_0^2$$ compare 11.47.

$$\omega^2 \longrightarrow \frac{g(L_1+L_2)}{2L_1 L_2} \pm \sqrt{\frac{g^2 (L_1+L_2)^2}{4L_1^2 L_2^2} - \frac{g^2}{L_1 L_2}} =$$

$$= \frac{g}{2L_2} + \frac{g}{2L_1} \pm \frac{g}{2L_1 L_2} \sqrt{L_1^2 + L_2^2 + 2L_1 L_2 - 4L_1 L_2} =$$

$$= \frac{g}{2L_2} + \frac{g}{2L_1} \pm \frac{g}{2L_1 L_2} (L_1 - L_2) = \frac{g}{2L_2} + \frac{g}{2L_1} \pm \left(\frac{g}{2L_2} - \frac{g}{2L_1}\right)$$

or $\omega^2 = \dfrac{g}{L_2}$ or $\boxed{\omega^2 = \dfrac{g}{L_1}}$

(11.17) $m_1 = 8m$; $m_2 = m$; $L_1 = L_2 = L$

$$\omega^2 = \frac{m_1+m_2}{2m_1} g \frac{(L_1+L_2)}{L_1 L_2} \pm \sqrt{\frac{(m_1+m_2)^2}{4m_1^2} g^2 \frac{(L_1+L_2)^2}{L_1^2 L_2^2} - \frac{m_1+m_2}{m_1 L_1 L_2}}$$

$$\Rightarrow \omega^2 = \frac{9g}{8L} \pm \sqrt{\frac{9^2 g^2}{8^2 L^2} - \frac{9g^2}{8L^2}} = \frac{9g}{8L} \pm \frac{9g}{8L}\sqrt{1 - 8/9} =$$

$$= \frac{9g}{8L}\left(1 \pm \frac{1}{3}\right) = \frac{3}{4}\omega_0^2; \quad \frac{3}{2}\omega_0^2$$

$$\begin{pmatrix} 9mgL - \omega^2 9mL^2 & -\omega^2 mL^2 \\ -\omega^2 mL^2 & mgL - \omega^2 mL^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 9\omega_0^2 - 9\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0$$

$$9(\omega_0^2 - \omega^2)\varphi_1 - \omega^2 \varphi_2 = 0 \Rightarrow \varphi_2 = \frac{9(\omega_0^2 - \omega^2)}{\omega^2}\varphi_1$$

For $\omega^2 = \frac{3}{4}\omega_0^2$; $\varphi_2 = \frac{9(1/4)\omega_0^2}{\frac{3}{4}\omega_0^2}\varphi_1 = 3\varphi_1$

$$\Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\omega^2 = \frac{3}{2}\varphi_0^2$; $\varphi_2 = \frac{9\left(-\frac{1}{2}\omega_0^2\right)}{\frac{3}{2}\omega_0^2}\varphi_1 = -3\varphi_1$

$$\Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Don't forget the real part of the solution!

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = A\cos\omega_1 t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + B\cos\omega_0 t \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \dot\varphi_1 \\ \dot\varphi_2 \end{pmatrix}}_{\text{of } 0} = A\begin{pmatrix} 1 \\ 3 \end{pmatrix} + B\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \Rightarrow B = -A \text{ and } 3(A - B) = 3(2A) = \alpha$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \frac{\alpha}{6}\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}\cos\omega_1 t - \begin{pmatrix} 1 \\ -3 \end{pmatrix}\cos\omega_0 t\right]$$

where $\omega_1 = \sqrt{3}/2\,\omega_0$; $\omega_2 = \sqrt{3/2}\,\omega_0$

It is not periodical, since there is no T for which $\begin{pmatrix} \varphi_1(t+T) \\ \varphi_2(t+T) \end{pmatrix} = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$