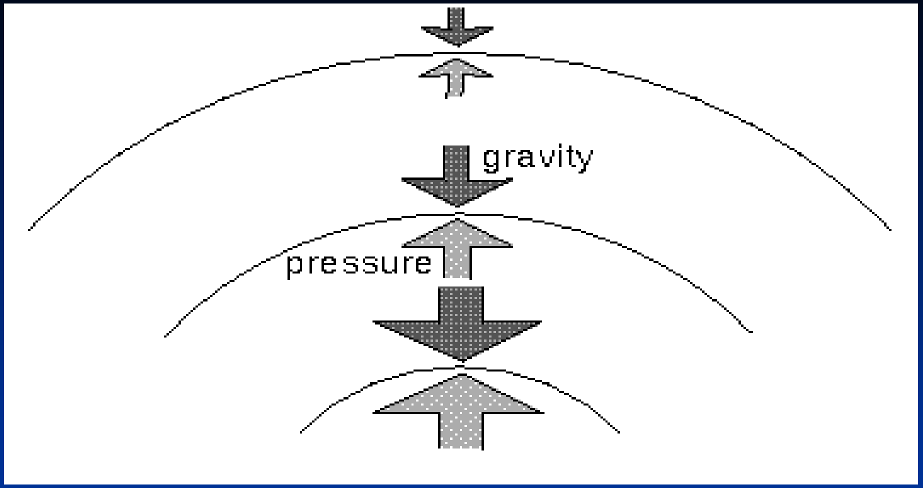
GRAVITATIONAL POTENTIALS

*definitions and Gauss Theorem
*density-potential pairs
*spherical potentials
*axisymmetric potentials
*triaxial potentials

- Gas: hydrostatic equilibrium



The downward gravitational force $\frac{G\,M(r)}{r^2} 4\pi r^2 \rho(r) dr$

Outward pressure force $4\pi r^2 \frac{dP}{dr} dr$

$$\frac{dP}{dr} = -\frac{G\,M(r)}{r^2} \rho(r)$$

r --- radius vector
M(r) --- mass within r
ρ(r) --- mass density
P(r) --- gas pressure at r

- Definitions: find force or potential field of a stellar distribution

Describe mass distribution as a continuous function

In a 1-D system: always possible to define potential energy U(x) corresponding to any given force f(x):

$$U(x) = -\int_{x_0}^{x} dx'\, f(x')$$

→ integral over closed path vanishes

where x_0 is arbitrary position at which U=0. The choice of x_0 does not affect the dynamics

Gravitational potential is the gravitational energy per unit mass

Hence, gravitational energy of mass m is $U(x) = m\,\Phi(x)$

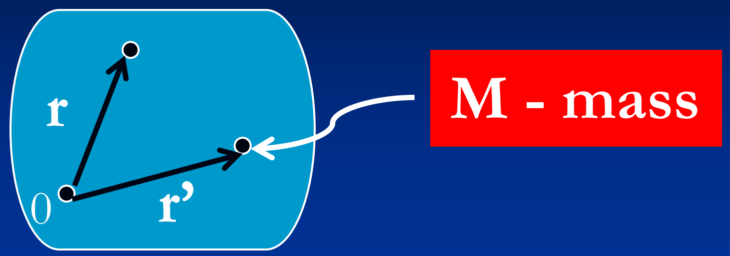
Note, that because U depends on the endpoints only:

$$f(x) = -\vec{\nabla} U$$

→ conservative field

In multi-dimensional space:

gravitational force: vector field



$$dM(r') = \rho\, d^3 r'$$

For an arbitrary density distribution:

$$d\Phi(r) = -G\, dM(r') / |r' - r|$$

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, d^3\mathbf{r}' \qquad (2\text{-}1a)$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\, \rho(\mathbf{r}')\, d^3\mathbf{r}' \qquad (2\text{-}1b)$$

Gauss Theorem (for gravity)

Remember: divergence of a vector

$$Divergence\ \ of\ \ \mathbf{A} = div\,\mathbf{A} = \vec{\nabla}\cdot\mathbf{A}$$

$$= (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Taking divergence of eq.(2-1b): $\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G\int_V \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\rho(\mathbf{r}')\, d^3\mathbf{r}'$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G\rho(\mathbf{r})$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$$ ← Poisson eq. (inside M)

$$\nabla^2\Phi(\mathbf{r}) = 0$$ ← Laplace eq. (outside M)

Note, in 1-D this is trivial (spherical):

$$dF = -G\, dM(r)/r^2 = -4\pi G\rho(r)dr$$

$$\nabla F = -4\pi G\rho(r) = -\nabla^2\Phi \equiv -\Delta\Phi$$

But in 3-D, you should remember that (gradient)

$$\nabla_r\left(\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right) = \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}$$

and (divergence):

product rule:

$$\nabla_r \cdot \left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\right) = -\frac{3}{|\mathbf{r}'-\mathbf{r}|^3} + \frac{3(\mathbf{r}'-\mathbf{r})\cdot(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|^5} = 0$$

cancels

when $\mathbf{r}' \neq \mathbf{r}$

So, to take the divergence of F(r): $\nabla_r \bullet F$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G\int_V \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\,\rho(\mathbf{r}')\,d^3\mathbf{r}'$$

$$\nabla_r \cdot \mathbf{F}(\mathbf{r}) = G\int_V \nabla_r \cdot \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\,\rho(\mathbf{r}')\,d^3\mathbf{r}'$$

So, the only contribution to $\nabla_r \bullet F$ comes from the point **r’ = r**

Take a small sphere with radius |**r’-r**|=*h* centered on this point:

$$\nabla_r \bullet F = G\rho(\mathbf{r})\int_{|\mathbf{r}'-\mathbf{r}|\le h} \nabla_{\mathbf{r}} \bullet \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} d^3\mathbf{r}' =$$

r → r’ ⟶

$$= -G\rho(\mathbf{r})\int_{|\mathbf{r}'-\mathbf{r}|\le h} \nabla_{\mathbf{r}'} \bullet \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} d^3\mathbf{r}' =$$

Divergence theorem: to replace volume integral by integral over enclosed surface

$$= -G\rho(\mathbf{r})\int_{|\mathbf{r}'-\mathbf{r}|=h} \frac{(\mathbf{r}'-\mathbf{r})\bullet \mathbf{d}^2\mathbf{S}'}{|\mathbf{r}'-\mathbf{r}|^3} =$$

$$= -G\rho(r)\int d^2\Omega = -4\pi G\rho(\mathbf{r})$$

Poisson eq.

where (on the surface): |r’-r|=h and $d^2\mathbf{S} = (\mathbf{r}'-\mathbf{r})h\, d^2\Omega$

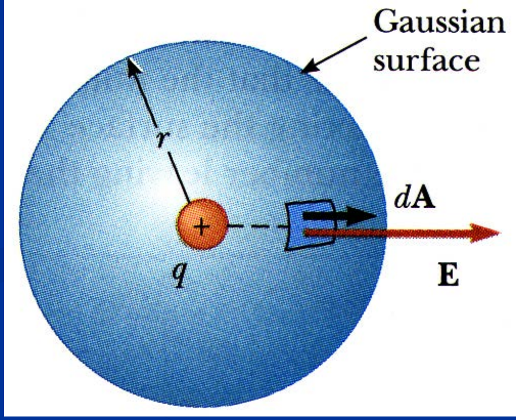
➢Application of Gauss Theorem

Note, relationship between Φ and ρ are *linear*

For volume V with a surface A enclosing mass M

$$\begin{aligned} 4\pi G M &= 4\pi G \int_V \rho(\mathbf{r})\, d^3\mathbf{r} \\ &= \int_V -\nabla \cdot \mathbf{F}(\mathbf{r})\, d^3\mathbf{r} = \int_A -\mathbf{F}(\mathbf{r}) \cdot d^2\mathbf{S} \end{aligned} \quad (2\text{-}2)$$

application of Gauss theorem



The potential energy W of self-gravitating system can be defined by setting Φ = 0 at infinity, and

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\, \Phi(\mathbf{r})\, d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\, d^3\mathbf{r}$$

So, $W < 0$ always

➢ Density-potential pairs

Consider potential for an arbitrary spherical mass distribution:

$$\Phi(r) = -\int_{r_0}^{r} dr'\, a(r) = G\int_{r_0}^{r} dr' \frac{M(r')}{r'^2}, \quad with\ r_0 = \infty$$

→ Φ (r) < 0 everywhere

with the enclosed mass

$$M(r) = 4\pi \int_{r_0}^{r} dr'\, r'^2 \rho(r')$$

❖ Point mass (Keplerian potential)

$\Phi(r) = - GM/r$

$$F(r) = -\nabla\Phi = d\Phi/dr = -GM/r^2$$

$v_c^2(r) = GM/r = - \Phi(r)$ circular velocity

$v_{esc}^2(r) = 2GM/r = - 2\Phi(r)$ escape velocity

❖ Uniform spherical shell

Outside: $\Phi(r) = - GM/r$ (Keplerian)

Inside: $\Phi(r) = const.$; $F(r) = 0$

❖ Homogeneous (uniform) sphere

$$\rho(r) = \begin{cases} \text{const} & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

outside: $\Phi(r) = -GM/r$ (Keplerian)

inside: $\Phi(r) = -2\pi G\rho\,(a^2 - r^2/3)$

$$F_r = -\,GM(r)/r^2 = -\,(4/3)\,\pi G\rho r$$

harmonic oscillator

So, $k/m = (4/3)\,\pi G\rho = \omega^2$ and $P_r = 2\pi/\omega$

radial period of oscillations $P_r = (3\pi/G\rho)^{1/2}$

and free-fall time $t_{ff} \sim (1/4)\,P_r \sim (G\rho)^{-1/2}$

Because $F_r = v^2/r$ → $v_c(r) = \omega\, r = [(4/3)\,\pi G\rho]^{1/2} r$

We define $\Omega(r) = \omega$ (= const in this case) ➔ solid body rotation

Note that $P_c = P_r$

Logarithmic potential

We know that many rotation curves are flat at large radii, $v_c \sim v_0$, so

$$\Phi(r) = G\int_{r_0}^{r} dr' \frac{M(r')}{r'^2}, \quad with\ r_0 = \infty$$

$$\Phi(r) = -V_0^2 \ln r + const$$

meaning that potential behaves as logarithmic…

Spherical systems

For power law: $\rho = \rho_0 (r/a)^{-\alpha}$ we have:

$$\Phi(r) = -[(4\pi G a\rho_0)/(3-\alpha)]\ (r/a)^{2-\alpha} = v_c^2/(\alpha-2)$$

- for $\alpha > 3$, $M(<r) \rightarrow$ infinity for $r \rightarrow 0$ $\rightarrow$ infinite mass at the origin

•for $\alpha = 2$, we have singular isothermal sphere with circular velocity

$$v_c(r) = (4\pi\, G\, a^2\ \rho_0)^{1/2} = \text{const.}\ \text{ at all radii,}$$

yielding $\Phi(r) = 4\pi G a^2 \rho_0 \ln(r/a)$

❖More specific spherical models

•Hernquist

$$\rho_H(r) = \frac{Ma}{2\pi r(r+a)^3} \quad ; \quad \Phi_H(r) = -\frac{GM}{(r+a)}$$

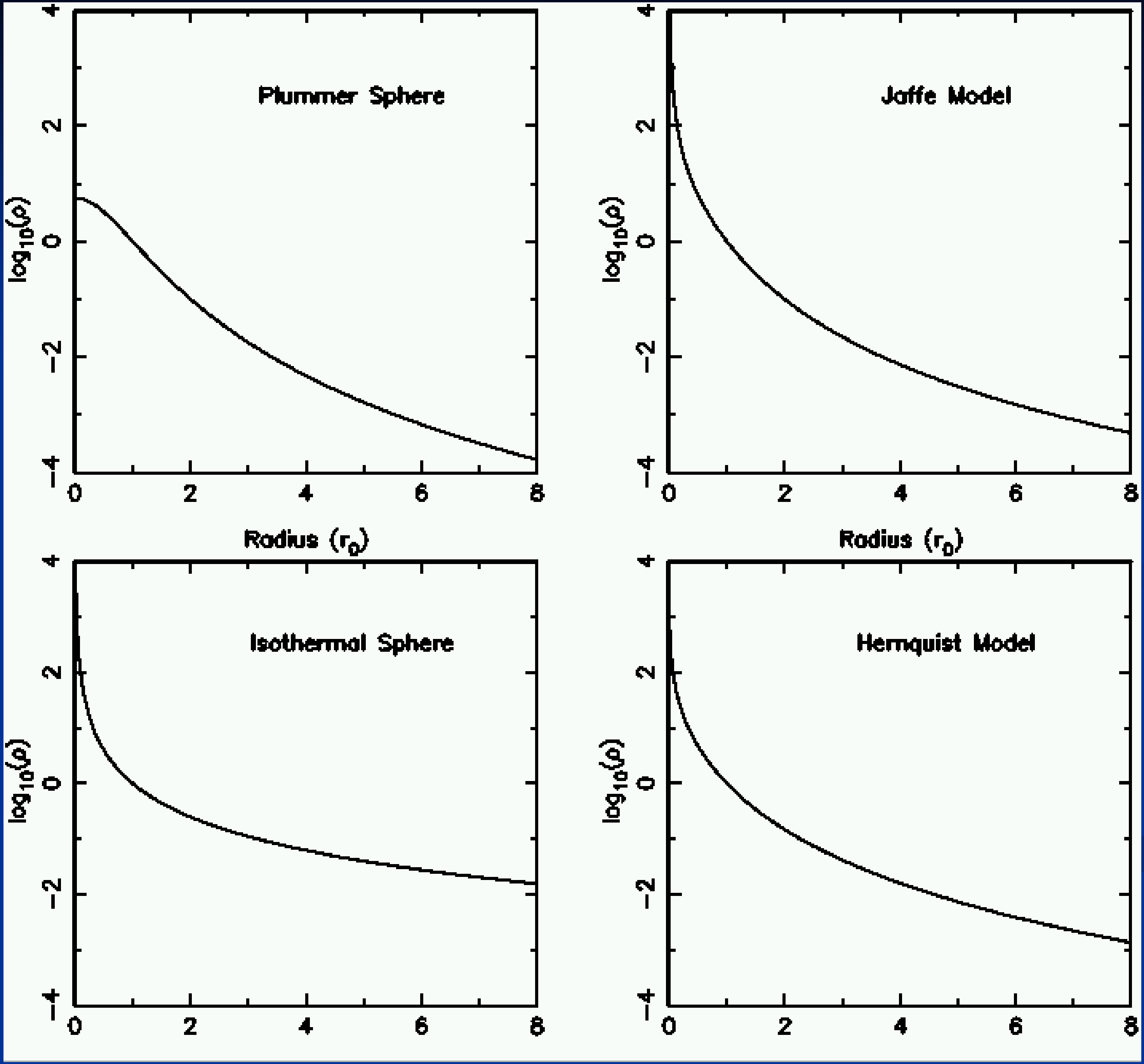
•Jaffe

$$\rho_J(r) = \frac{Ma}{4\pi r^2(r+a)^2} \quad ; \quad \Phi_J(r) = -\frac{GM}{a}\ln\left(\frac{a}{r+a}\right)$$

•Plummer sphere

$$\rho_P(r) = \left(\frac{3M}{4\pi b^3}\right)\left(1+\frac{r^2}{b^2}\right)^{-5/2} \quad ; \quad \Phi_P(r) = -\frac{GM}{\sqrt{r^2+b^2}}$$

ρ(R) for various spherical models



❖ Axisymmetric thin disks (cylindrical r, z)

- Vertical (z) potential near the plane $z = 0$:

within the disk: ρ_0 the volume density at the $z = 0$ plane

above the disk: surface density $\Sigma(z)$

$$4\pi GM = 4\pi G \int_V \rho(\mathbf{r})\, d^3\mathbf{r} = \int_V -\nabla \cdot \mathbf{F}(\mathbf{r})\, d^3\mathbf{r} = \int_A -\mathbf{F}(\mathbf{r}) \cdot d^2\mathbf{S}$$

Using the Gauss theorem, eq.(2-2): $\nabla^2\Phi$ → $\nabla\Phi$

$$-\frac{\partial \Phi}{\partial z} = g_z = 4\pi G\rho_0 z = 2\pi G\Sigma(z) \quad \text{(inside)}$$
$$= 2\pi G\Sigma \quad \text{(above)}$$

where $\Sigma = \int_0^\infty dz\, \Sigma(z)$

Note, unlike spherical potentials,
disk potential depends on the mass outside r

❖Examples:

•Mestel disk: $\Sigma(r) = \Sigma_0 \, r/r_0$ has $v_c^2(r) = 2\pi G \Sigma_0 r_0 = GM(<r)/r$

unusual case when v_c is independent of $M(>r)$!

•Exponential disk: $\Sigma(r) = \Sigma_0 \, e^{-r/r_d}$

fits the light profile in a much more realistic way than Mestel disk, and has circular velocity (see analytical approximation we used!):

$$V_c^2(R) \; = \; 4\pi G \, \Sigma_0 \, R_d \, y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y) \right]$$

where $y = r/2r_d$, and

I_n, K_n are Bessel functions of the 1st and 2nd kind

•Kuzmin-Toomre disk:

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}} \; ; \quad \Phi_K(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}$$

Note, because Poisson equation is linear in ρ, Φ:
differences between density-potential pairs and
differentials of density-potential pairs are *also* ρ–Φ pairs

•Toomre disk sequence: of order *n* can be obtained from the above Kuzmin disks by differentiation with respect to a^2:

$$\Sigma_{T_n}(R) = \left(\frac{d}{da^2}\right)^{n-1} \Sigma_K(R) \; ; \quad \Phi_{T_n}(R) = \left(\frac{d}{da^2}\right)^{n-1} \Phi_K$$

Here n=1 Kuzmin disk; n=infinity is Gaussian disk

•Bessel disk:

$$\Sigma(r) = \frac{k}{2\pi G} J_0(kr); \qquad \Phi(r, z) = \exp(-k|z|) J_0(kr)$$

❖Axisymmetric flattened systems

Realistic bulge + disk, etc. systems are neither spherical nor thin disks

Combining both we get *flattened* potentials

- Miyamoto-Nagai flattened system:

$$\rho_M(R,z) = \left(\frac{Mb^2}{4\pi}\right)\frac{aR^2+(a+3B)(a+B)^2}{[R^2+(a+B)^2]^{5/2}B^3}$$

$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2+(a+B)^2}} \quad ; \quad B^2 = z^2+b^2$$

If a = 0, we get Plummer sphere,
and
if b = 0, we get Kuzmin disk

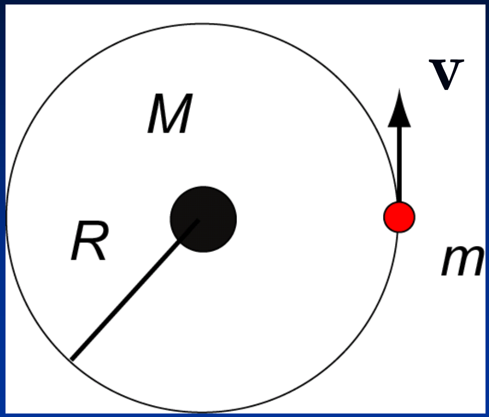
THE VIRIAL THEOREM

*illustrations
*general case
*mass determination
*binding energy
*specific heat: gravothermal catastrophe

Illustrations

Circular orbits

Consider the mass m in a circular orbit around M ($>> m$)



$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

Multiply by *r*:

$$mv^2 = \frac{GmM}{r} \quad ===> \quad 2K = -W \quad \text{or} \quad 2K + W = 0$$

2K

-W

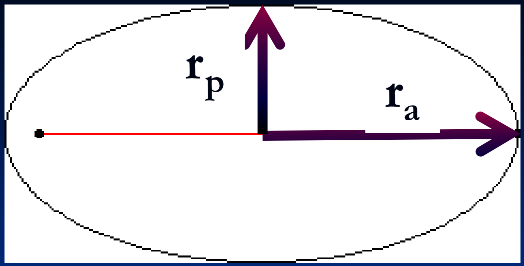
Define the ratio $\eta = K/|W| = 1/2$

E = -K, where

E = K+W → the total energy

Note, that in *this case* instantaneous value is also time-averaged value

- Time-averaged Keplerian orbit (elliptical orbits)



In general, η changes along the Keplerian orbit

Example: compare the pericentric, η_p, and apocentric, η_a, values:

$$\frac{\eta_p}{\eta_a} = \frac{v_p^2 r_p}{v_a^2 r_a} = \frac{r_a}{r_p} \neq 1$$

using $r_p v_p = r_a v_a$ (angular momentum conservation)

Taking time averages over an orbit:

$$\langle -W \rangle = \langle GM/r \rangle = GM/\langle r \rangle$$

and $\langle K \rangle = \langle 0.5 v^2 \rangle = GM/2\langle r \rangle$

$\Rightarrow$ $\eta = 0.5$, $E = -K$ (as before)

Note, that time averages for a single non-Keplerian orbit do *not* usually have $\eta=0.5$. But this always holds when averaged over all the particles. Above, *m* and *M* form the whole system, with K=0 for *M*.

General case

Consider a cluster of N stars with time-dependent potential $\Phi(\mathbf{r}, t)$. Individual energies are not conserved but the total E is. To show this, we write the 2nd law of Newton:

$$\frac{d}{dt}(m_i \mathbf{v}_i) = -\sum_{\substack{j \\ i\neq j}} \frac{G m_i m_j}{|\mathbf{r_i} - \mathbf{r_j}|^3}(\mathbf{r_i} - \mathbf{r_j}) \qquad (2\text{-}3)$$

→ m_i cancels out

Next, take the scalar product of this equation with $\mathbf{v_i}$:

$$\sum_i \mathbf{v_i} \cdot \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{d}{dt}K = -\sum_{\substack{i,j \\ i\neq j}} \frac{G m_i m_j}{|\mathbf{r_i} - \mathbf{r_j}|^3}(\mathbf{r_i} - \mathbf{r_j}) \cdot \mathbf{v_i} \qquad (2\text{-}4)$$

Repeating the same procedure with a star $\mathbf{v}_j$:

$$\frac{1}{2}\sum_j \mathbf{v}_j \cdot \frac{d}{dt}(m_j \mathbf{v}_j) = -\sum_{\substack{i,j \\ i\neq j}} \frac{G m_i m_j}{|\mathbf{r_i} - \mathbf{r_j}|^3}(\mathbf{r}_j - \mathbf{r}_i) \cdot \mathbf{v}_j \qquad (2\text{-}5)$$

Adding the right-hand sides of eqs.(2-4) and (2-5) →

Adding the right-hand sides of eqs.(2-4) and (2-5)

$$-\sum_{\substack{i,j\\ i\neq j}} \frac{Gm_i m_j}{\left|\mathbf{r_i}-\mathbf{r_j}\right|^3}(\mathbf{r}_i-\mathbf{r}_j)\cdot(\mathbf{v_i}+\mathbf{v}_j) = -\sum_{\substack{i,j\\ i\neq j}} \frac{d}{dt}\left(\frac{Gm_i m_j}{\left|\mathbf{r}_i-\mathbf{r_j}\right|}\right).$$

This is equal to 2W:

$$\mathrm{W} = -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{\mathrm{Gm_i m_j}}{\left|\mathbf{r_i}-\mathbf{r_j}\right|} = \frac{1}{2}\sum_{\mathrm{i}} \mathrm{m_i}\,\Phi(\mathbf{r_i}) \quad \text{or} \quad \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r})\mathrm{dV}.$$

Note:
division by 2 means that
each pair will contribute one term only to the sum

Adding eqs.(2-4) and (2-5):

$$2\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{K}-\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{\mathrm{Gm_i m_j}}{\left|\mathbf{r_i}-\mathbf{r_j}\right|}\right) = 0. \qquad (2\text{-}6)$$

→ E = K + W = const

According to eq.(2-6):

the stars in an isolated cluster can change their kinetic and potential energies, as long as their sum remains constant

The Virial Theorem:

on average, the kinetic and potential energies are in a specific balance

$$\frac{d}{dt}(m_i \mathbf{v}_i) = -\sum_{\substack{j \\ i\neq j}} \frac{G m_i m_j}{\left|\mathbf{r_i} - \mathbf{r_j}\right|^3}(\mathbf{r_i} - \mathbf{r_j})$$

Proof:

Start again with eq.(2-3), with an addition of an external force $\mathbf{F_{ext}}$.

Next, take scalar product with $\mathbf{r_i}$ and sum over all stars:

$$\sum_{\mathbf{i}} \frac{d}{dt}(m_i \mathbf{v_i}) \cdot \mathbf{r_i} = -\sum_{\substack{\mathbf{i,j} \\ \mathbf{i\neq j}}} \frac{G m_i m_j}{\left|\mathbf{r_i} - \mathbf{r_j}\right|^{\mathbf{3}}}(\mathbf{r_i} - \mathbf{r_j}) \cdot \mathbf{r_i} + \sum_{\mathbf{i}} \mathbf{F^i_{ext}} \cdot \mathbf{r_i}. \qquad (2\text{-}7)$$

A similar equation would result if we started with the j-force:

$$\sum_{j}\frac{d}{dt}(m_j \mathbf{v_j})\cdot\mathbf{r_j} = -\sum_{\substack{i,j\\ i\neq j}}\frac{Gm_i m_j}{\left|\mathbf{r_i}-\mathbf{r_j}\right|^3}(\mathbf{r_j}-\mathbf{r_i})\cdot\mathbf{r_j} + \sum_{j}\mathbf{F_{ext}^{j}}\cdot\mathbf{r_j}. \qquad (2\text{-}8)$$

The left sides of these two equations are the same; each equal to

$$(rr)'' = 2(rr'' + r'r')$$

$$\frac{1}{2}\sum_{i}\frac{d^2}{dt^2}(m_i\mathbf{r}_i\cdot\mathbf{r}_i) - \sum_{i} m_i\mathbf{v}_i\cdot\mathbf{v_i} = \frac{1}{2}\frac{d^2 I}{dt^2} - 2K,$$

where ***I*** is the moment of inertia of the system:

$$I \equiv \sum_{i} m_i\mathbf{r}_i\cdot\mathbf{r_i}$$

Averaging eqs.(2-7) and (2-8): the first term on the right-hand side is the potential energy **W**, so

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2K = W + \sum_{i}\mathbf{F_{ext}^{i}}\cdot\mathbf{r_i} \qquad (2\text{-}9)$$

Taking long-term average of eq.(2-9) over time interval $0 < t < \tau$:

$$\frac{1}{2\tau}\left[\frac{dI}{dt}(\tau)-\frac{dI}{dt}(0)\right] = 2 < K > + < W > + \sum_i < \mathbf{F}^{i}_{ext} \cdot \mathbf{r}_i > . \qquad (2\text{-}10)$$

As long as the stars are bound to the cluster, the products $|\mathbf{r}_i \,.\, \mathbf{v}_j|$, and hence $|dI/dt|$, never exceed some finite limits

Thus, the left-hand side of eq.(2-10) must tend to zero as $\tau \to \infty$, giving:

$$2 < K > + < W > + \sum_i < \mathbf{F}^{i}_{ext} \cdot \mathbf{r}_i > = 0.$$

the Virial Theorem

Note: one can distinguish two types of kinetic energy:

$$K_{i,j} = \int \tfrac{1}{2}\,\rho\,\langle v_i v_j \rangle\, d^3r$$ -- total K

$$T_{i,j} = \int \tfrac{1}{2}\,\rho\,\langle v_i \rangle \langle v_j \rangle\, d^3r$$ -- ordered motion

$$\Pi_{i,j} = \int \rho\,\sigma^2_{i,j}\, d^3r$$ -- random motion

Reviewing conditions for Virial Theorem:

- The system must be self-gravitating
- The system must be in steady state:
  orbital timescale << evolution timescale
- Quantities must be time-averaged
  (or many objects sampled with random orbital phase)
- The system must be isolated,
  or at least embedded in a slowly varying potential
- The system can be collisionless (stars) or collisional (gas)

ALSO: when the total energy is negative, the self-gravitating system is bound

$$E = K + W = -K = \frac{1}{2}W$$

Mass determination

The most interesting use of the virial theorem is mass determination of stellar systems

For a system of total mass *M* and mean squared velocity $\langle v^2 \rangle$:

$$\mathbf{K = 0.5\,M \langle v^2 \rangle}$$

↓

$$\langle \mathbf{v}^2 \rangle = -\mathbf{W/M} \equiv \mathbf{GM/r_g}$$

defines the gravitational radius $\mathbf{r_g}$

But stellar systems don’t have sharp edges!

Define “median radius” r_h which encloses half the mass.
For many systems $\mathbf{r_h \cong 0.4\, r_g}$, then

$$\mathbf{M_{tot}} \cong \frac{\langle \mathbf{v}^2 \rangle \mathbf{r_h}}{\mathbf{0.4G}}$$

Binding energy

System which is spread out and at rest has **E = K = W = 0**

After settling down (virializing): **E = K+W = -K**

↓

- Energy must be released during the gravitational collapse
- This energy is termed the binding energy – it is needed to unbind the system
- The value of the binding energy is equal to the remaining K
- The total gravitational energy released is **-W**, of which half goes into **K** and half escapes the system

Examples:

- Collapsing protostars are luminous: they radiate half of their gravitational potential energy

- Kelvin considered a gravitational origin of Sun's energy, via gradual contraction

- For a 'typical' galaxy: $\mathbf{K \sim 0.5\, M v_c^2} \sim 10^{57}$ ergs $\sim 10^{10}\, L_8 \times 10^7$ yrs

↓

this is $3\ 10^{-7}$ of the rest mass (this is negligible!)

➢Specific heat of self-gravitating systems

Define the temperature **T** of self-gravitating system (of **N** stars) by analogy with the ideal gas

$$\frac{1}{2}\mathbf{m} < \mathbf{v}^2 > = \frac{3}{2}\mathbf{k}_B\mathbf{T}$$

Note: $<v^2>=3\sigma^2$

where **m** is the stellar mass

$\mathbf{k_B}$ is the Boltzmann constant

We use spatially averaged $\mathbf{v^2}$ and **T**, for example:

$$< T > \equiv \int \rho(\mathbf{r})\, T\, dV / \int \rho(\mathbf{r})\, dV$$

The total kinetic energy is then $\mathbf{K = (3/2)\, Nk_B<T>}$

Using virial theorem: $\mathbf{E = -K}$, and $\mathbf{E = -(3/2)\, Nk_B<T>}$.

The heat capacity of the system is $$\mathbf{C} \equiv \frac{\delta \mathbf{E}}{\delta < \mathbf{T} >} = -\frac{3}{2}\mathbf{N\kappa_B} < \mathbf{0}\ !!$$

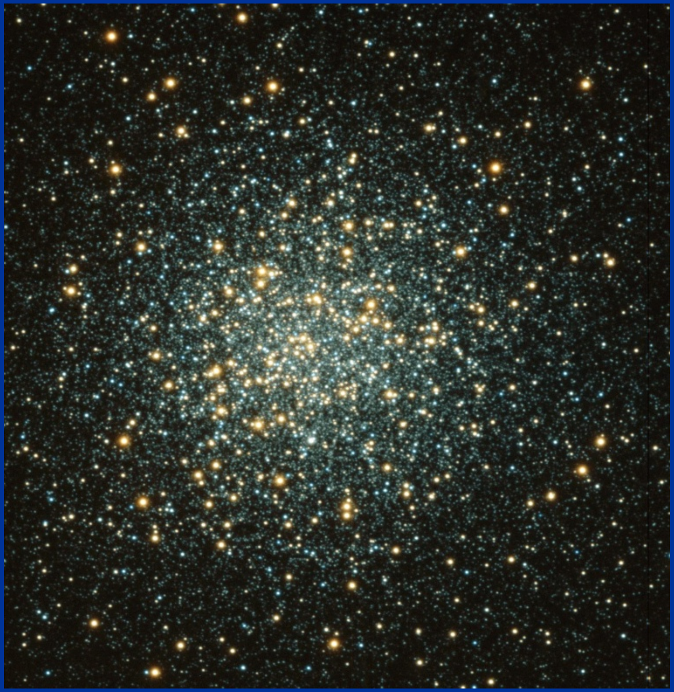
Note, by losing energy the system gets hotter!

❖Negative specific heat: by losing energy the system gets hotter

$$\Delta E < 0 \quad \Longleftrightarrow \quad \Delta K > 0$$

Energy decreasing

"Temperature" increasing



•Any self-gravitating bound system has a negative heat capacity:

stars, stellar clusters, galaxies, galactic clusters, etc.

•Thermodynamically, such systems exhibit counter-intuitive behavior

Example:
a bound self-gravitating system in contact with a heat bath

•Initially: thermal equilibrium at **T**. How stable is this equilibrium?

Note: $\langle v^2 \rangle = 3\sigma^2$ for isothermal sphere

•By transferring a small amount of heat **dQ > 0** to the bath, the stellar system will change to $\mathbf{T - dQ/C = T + dQ/|C|}$

•The stellar system is now hotter than the bath and heat continues to flow from hot (system) to cold (bath)

•Such system is thermally unstable and experiences a thermal runaway

❖ Gravothermal catastrophe:

Antonov (1962)
Lynden-Bell & Wood (1968)

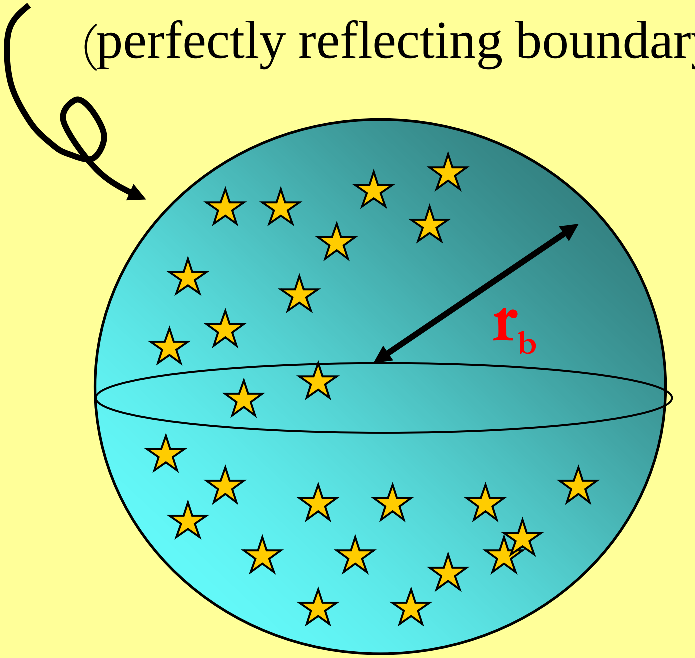
Consider:

Adiabatic wall

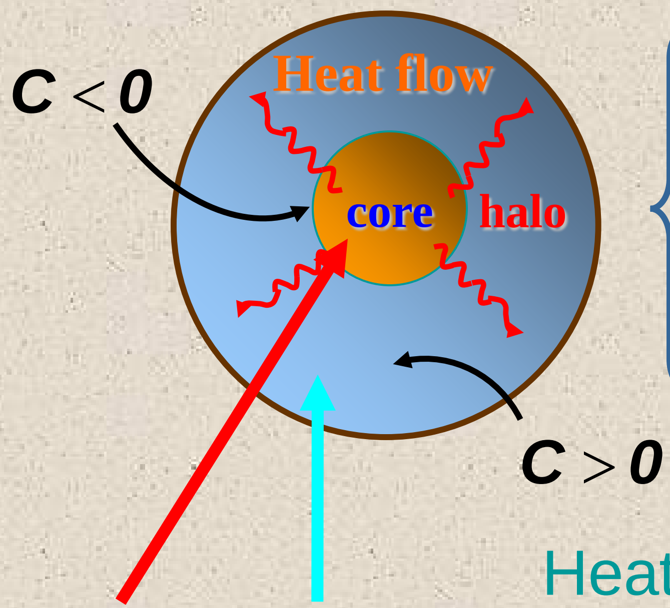
(perfectly reflecting boundary)

Self-gravitating N-body system

- **radius** : r_b
- **mass** : $M = N \times m$
- **energy** : E



Gravothermal Catastrophe



core = self-gravitating — Negative specific heat $C < 0$

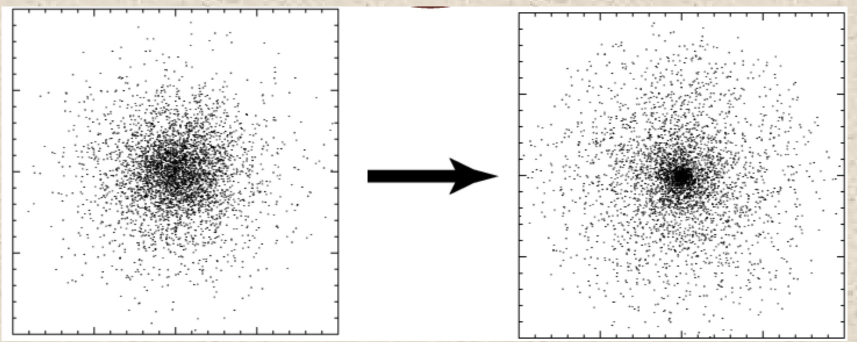
halo = normal system (heat bath) — Positive specific heat $C > 0$

$T_{core} > T_{halo}$ ⟹ Heat flow from core to halo ⟹ $\Delta T_{core} \uparrow$ $C < 0$, $\Delta T_{halo} \uparrow$

$r_e > r_b$ ⟹ $\Delta T_{core} > \Delta T_{halo}$ extended halo has large heat capacity

heat flow does not stop!!

Core-collapse !!



Onset of instability: heuristical approach:

Halo: $C_h>0$ since no strong self-gravity

Core: $C_c<0$ since confined by gravity

If sudden core heat up → $T_c>T_h$: heat flow from the core to the halo and the temperatures of BOTH rises

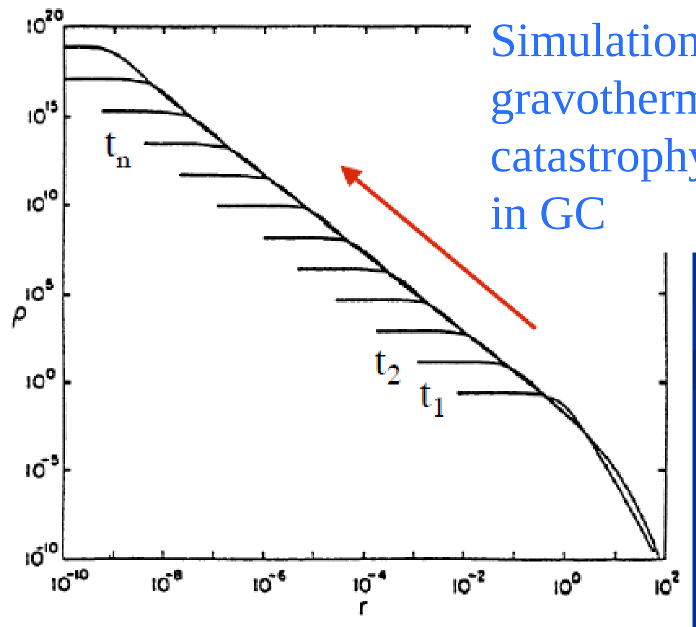
If $C_h<|C_c|$, $T_h = dQ/C_h$ rises faster than $T_c = dQ/C_c$ the heat flow shuts off

If $C_h>|C_c|$, T_h rises slower than T_c → the difference increases

The gravothermal instability sets in at R $(= \rho_c / \rho_{boundary}) = 708.61$

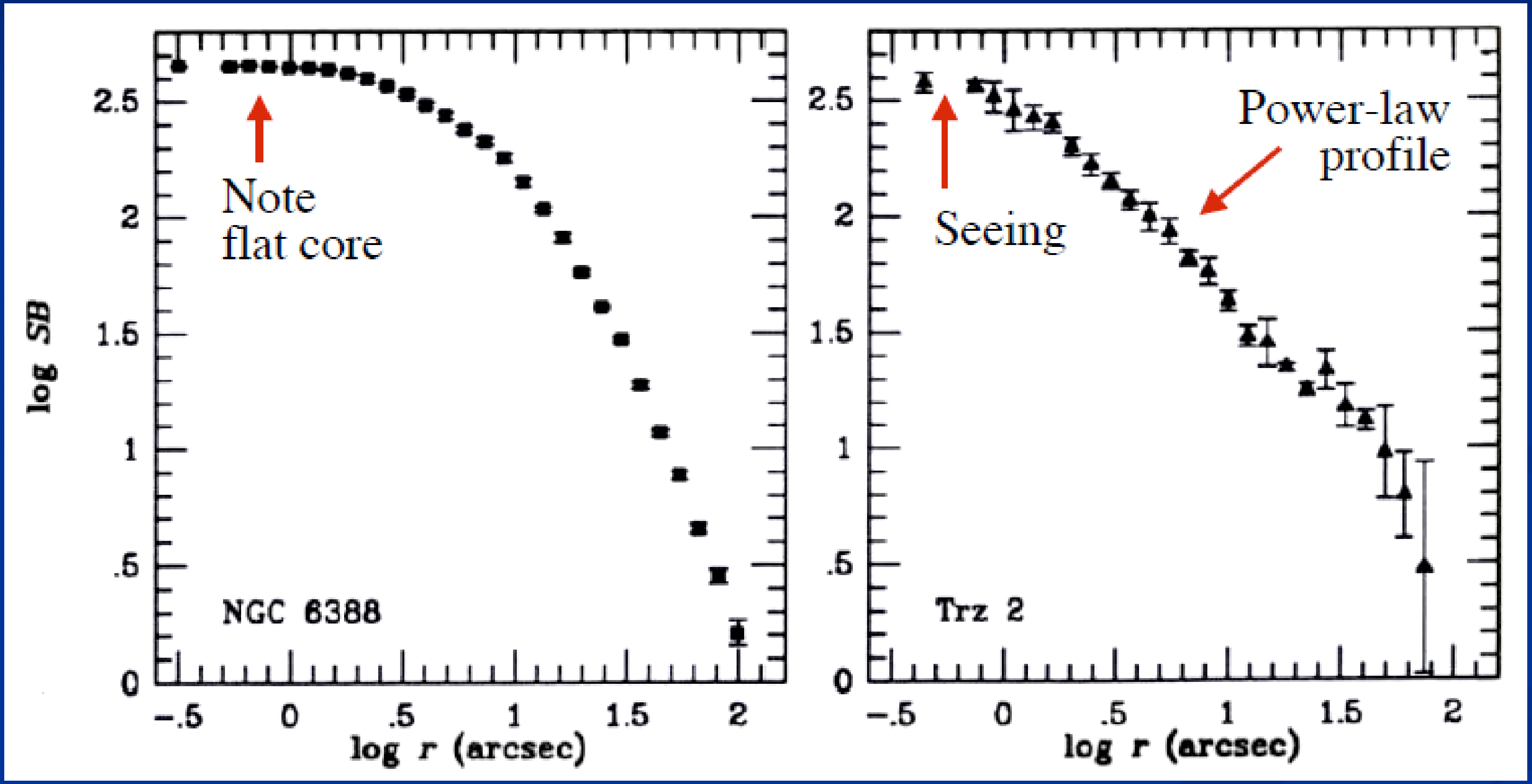
Is there an instability in the real systems (stars and gas)?

- The gravothermal catastrophe in a gas: develops through heat conduction, →growth time ~ thermal diffusion time
- In stellar systems: thermal diffusion is ~ relaxation time



Simulation of gravothermal catastrophy in GC

Surface brightness for globular clusters: evidence for core collapse



About 20% of globular clusters show cuspy cores