STELLAR-DYNAMICAL SYSTEMS

*distribution function
*collisionless Boltzmann (Vlasov) equation
*Jeans equations and applications

NOTE: we first consider **collisionless dynamics**:

- Definition of a collision: here we mean star-star deflection, not a direct impact

- For a collisionless case: stars are assumed to move in a completely smooth background potential

- For galaxies this almost always is a very good approximation

❖How to model motions of 10^{10} stars in a galaxy?

- Direct N-body approach (as in simulations)
  - At time t particles have $(m_i, x_i, y_i, z_i, vx_i, vy_i, vz_i)$, i=1,2,...,N (feasible for $N \ll 10^6$)
- Statistical or fluid approach (N very large)
  - At time **t** particles have a spatial density distribution n(x,y,z)*m,
  - at each point have a velocity distribution $G(v_x, v_y, v_z)$

The distribution function (DF)

- The system is fully described by its distribution function or phase space density:

$f(\mathbf{r}, \mathbf{v}, t)\, d^3\mathbf{r}\, d^3\mathbf{v}$ -- number of stars at **r** with **v** at time **t** in range $d^3\mathbf{r}$ and $d^3\mathbf{v}$

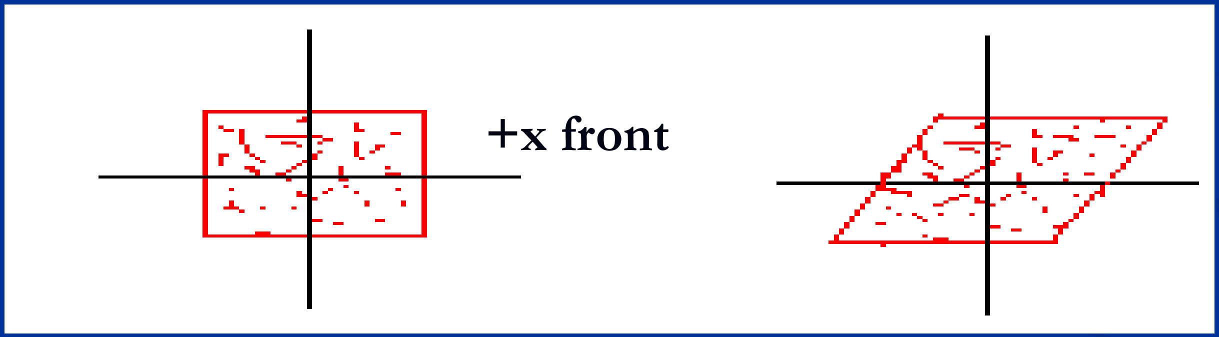
- Knowledge of the DF = holy grail, and gives a complete info about the system

- In practice: we observe $\boldsymbol{\Sigma}(\mathbf{r})$, $v(\mathbf{r})$, $\boldsymbol{\sigma}(\mathbf{r})$

- Essentially impossible to recover the DF from observations: one constraint: $\mathbf{f(r,v,t) > 0}$, as stars exists!

❖Phase space of stars (2-D example)

- N identical particles moving in a small bundle in phase space (Volume $=\Delta_x \Delta_v$)
- phase space deforms but maintains its area



- Gap widens between faster and slower stars
  - but the phase volume and number of stars are constants

❖Liouville theorem

- Phase space density of a group of stars is constant

  $f = N / \Delta x \, \Delta v_x = \text{const.}$

➢Collisionless **Boltzmann** (**Vlasov**) equation

- Continuity equation:

  stars are not created/destroyed since the flow preserves the number of stars

  stars do not jump across the phase space

  view the DF as a moving fluid of stars in 6-D space (**r**,**v**)

- Consider a 1-D example using **x** and $\mathbf{v_x}$ and remember that f is a number density

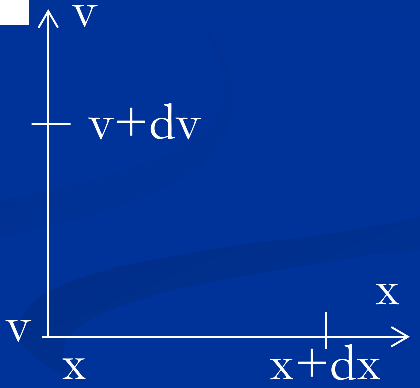
  focus on a small element of phase space $\mathbf{dx\ dv_x}$

•During dt, the net flow (change) along **x** is:

$$v_x dt\, dv_x \left[f(x, v_x, t) - f(x + dx\,, v_x, t)\right] \;=\; -\, v_x\, dt\, dv_x \frac{\partial f}{\partial x}\, dx$$

the net flow (change) due to the velocity gradient (along **v**):

$$dx\, \frac{dv_x}{dt} dt\, \left[f(x\,, v_x\,, t) - f(x\,, v_x + dv_x, t)\right] \;=\; -\, dx\; dt\, \frac{dv_x}{dt}\, \frac{\partial f}{\partial v_x} dv_x$$



the sum of these two equations is equal to the net change in *f*

$$dx\; dv_x\, \frac{\partial f}{\partial t} dt \;=\; -\, dt\, dx\; v_x\, \frac{\partial f}{\partial x}\, dv_x - dx\; dt\, \frac{dv_x}{dt}\, \frac{\partial f}{\partial v_x}\, dv_x$$

dividing by dx dv_x dt

$$\frac{\partial f}{\partial t} + v_x\, \frac{\partial f}{\partial x} + \frac{dv_x}{dt}\, \frac{\partial f}{\partial v_x} \;=\; 0$$

but since $$\frac{dv_x}{dt} = a_x = -\frac{\partial \Phi}{\partial x}$$

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v_x} = 0 \qquad (3\text{-}1)$$

adding **y** and **z** dimensions, we arrive at collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0 \qquad (3\text{-}2)$$

eq.(3-2) describes how DF changes with time as a result of:

conservation of stars
stars follow smooth orbits
flow of stars through **r** defines implicitly **v (=dr/dt)**
flow of stars through **v** is given explicitly by $-\nabla\Phi(\mathbf{r})$

Since $\partial f / \partial t$ as an *Eulerian* (partial) differential, it describes change in DF at a point in phase space

Now, consider Lagrangian (total or convective) differential, Df/Dt = df/dt

This describes the change in *f* as we follow along the 'orbit' through phase space

This Lagrangian derivative is nothing else but the left-hand side of the collisionless Boltzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} = 0$$

The phase space density *f* along the stellar orbit is constant

The flow is incompressible in phase space (Louville theorem)

For example:

if the region gets denser, σ (v-dispersion) will increase

if the region expands, σ will decrease

marathon race: starts --- *n* high, Δv high

ends --- *n* low, Δv low

➢The Jeans equations

- The collisionless Boltzmann equation is of a limited use:

  the constraints it provides are insufficient to find $f(\mathbf{r,v,t})$

  the complexity of f makes it observationally inaccessible

- What we observe are:

  mean velocities $\langle \mathbf{v} \rangle$

  velocity dispersions σ (which are $\langle \mathbf{v}^2 \rangle = \boldsymbol{\sigma}^2 + \langle \mathbf{v} \rangle^2$)

  stellar densities n or ρ

Need to rewrite Boltzmann equation in terms of these quantities

These quantities are ‘hidden’ within f → need to extract them by taking averages or moments

New variables:

$$number\ \ density: \quad n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)\, d^3\mathrm{v}$$ → 0th momentum in v

$$mean\ \ velocity: \quad < v_i(\mathbf{r},t) > = (1/n)\int \mathrm{v}_i\ f(\mathbf{r},\mathbf{v},t)\, d^3\mathrm{v}$$ → 1st momentum in v

❖Example:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v_x} = 0 \qquad (3\text{-}1)$$

Integrate the 1-D collisionless Boltzmann equation (3-1) over $\mathbf{v_x}$:

0th momentum

$$\frac{\partial n}{\partial t} + \frac{\partial (n\langle v_x\rangle)}{\partial x} = 0$$

continuity eq.
star number

Fluids: 1-D for comparison

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

where n = n(**x**,t) is the space density
$\langle \mathbf{v_x} \rangle$ drift velocity along **x**-axis

Multiply eq.(3-1) by $\mathbf{v_x}$ and integrate over $\mathbf{v_x}$,
insert the 0th momentum, rearrange terms:

1st momentum

$$\frac{\partial \langle v_x\rangle}{\partial t} + \langle v_x\rangle \frac{\partial \langle v_x\rangle}{\partial x} = -\frac{\partial \Phi}{\partial x} - \frac{1}{n}\frac{\partial (\overline{n\sigma_x^2})}{\partial x}$$

where $\boldsymbol{\sigma}_\mathbf{x}^2$ is the velocity dispersion
about mean velocity:
$\langle v_x^2\rangle = \langle v_x\rangle^2 + \sigma_x^2$

Repeating this in 3-D requires a little more care (see BT 4.2):

Jeans equation for coordinate *j*

$$\frac{\partial\langle v_j\rangle}{\partial t} + \langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i} = -\frac{\partial\Phi}{\partial x_j} - \frac{1}{n}\frac{\partial(n\sigma_{i,j}^2)}{\partial x_i} \qquad (3\text{-}3)$$

where summation convention applies: summation over repeated indexes

i = 1,2,3 and j = 1,2,3 refer to **x,y,z** (e.g., $x_2 = y$, $v_2 = v_y$, etc)

$$\sigma_{ij}^2 \equiv \overline{(v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)} = \overline{v_i\, v_j} - \langle v_i\rangle\langle v_j\rangle$$

This tensor is symmetric (check the righ hand side)

→ can be diagonalized → $\sigma_{ij}^2 = \sigma_{ii}^2\ \delta_{ij}$

This Jeans equation is similar to 2nd Newton's law d**v**/dt = **F**/m with:

left-hand side is the derivative of **<v>**

right-hand side contains force terms

❖Compare this to Euler's eq. for fluid flow

Euler for fluid:

Fluids: 1-D for comparison

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Phi - \frac{1}{\rho}\nabla p$$

Jeans for stars eq.(3-3)

$$\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} - \frac{1}{n}\frac{\partial (n\sigma_{i,j}^2)}{\partial x_i}$$

completely analogous!

- $n\sigma_{ij}^2$ is a stress tensor which clearly plays the role of anisotropic pressure
- In a fluid, pressure is scalar and therefore always isotropic for stellar 'fluid,' this tensor can be anisotropic σ_{ij} is symmetric tensor, and if $\sigma_{11} = \sigma_{22} = \sigma_{33}$ then the pressure is isotropic → Jeans and Euler eqs. are identical
- For stellar systems there is no equation of state linking pressure σ_{ij}^2 to ρ

Applications of the Jeans equations

- Deriving M/L profiles in spherical galaxies
- Deriving the flattening of a rotating spheroids with anisotropic velocity dispersion
- Analysis of asymmetric drift
- Surface (and volume) density in the galactic disk
- Analysis of the local velocity dispersion

Spherically-symmetric steady state systems

- Using Jeans eq.(3-3)

$$\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} - \frac{1}{n} \frac{\partial (n \sigma_{i,j}^2)}{\partial x_i}$$

Steady state: the 1st term is equal zero

Symmetry: $\langle v_r \rangle = \langle v_\theta \rangle = 0 \rightarrow \langle v_r^2 \rangle = \sigma_r^2$ and $\langle v_\theta^2 \rangle = \sigma_\theta^2$

In spherical polar coordinates, the steady state Jeans equation is (for spherically-symmetric stellar system):

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + \frac{1}{r}\left[2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\phi^2)\right] - \frac{\langle v_\phi \rangle^2}{r} = -\frac{d\Phi}{dr}$$

Introducing the anisotropy parameters

$$\beta_\vartheta = 1 - \sigma_\vartheta^2 / \sigma_r^2$$
$$\beta_\phi = 1 - \sigma_\phi^2 / \sigma_r^2$$

Replacing $\beta_\vartheta + \beta_\phi$ by 2β and v_{rot} for $< v_\phi >$:

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + 2\beta\frac{\sigma_r^2}{r} - \frac{V_{rot}^2}{r} = -\frac{d\Phi}{dr} \qquad (3\text{-}4)$$

which is equivalent to the equation of hydrostatic support:

dp/dr + anisotropic correction + centrifugal correction = F_{grav}

•Going further:

$$d\Phi/dr \rightarrow GM(<r)/r^2 = v_c^{\,2}/r$$

v_c – circular velocity

And re-writing the 1st term in eq.(3-4)

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + 2\beta\frac{\sigma_r^2}{r} - \frac{V_{\rm rot}^2}{r} = -\frac{d\Phi}{dr}$$

in logarithmic gradients:

$$V_{\rm rot}^2 - \sigma_r^2\left(\frac{d\ln n}{d\ln r} + \frac{d\ln(\sigma_r^2)}{d\ln r} + 2\beta\right) = \frac{GM(<r)}{r} = V_c^{\,2}$$

The last equation is similar to hydrostatic support in ideal gas with p = nkT

The equivalencies are:

$$\sigma_r^{\,2} \equiv T$$
$$d(\ln n)/d(\ln r) + d(\ln T)/d(\ln r) \equiv (n/p)\,dp/dr$$

and where 2β and $v_{rot}^{\,2}/r$ are anisotropy and rotation corrections