Stellar Encounters

*encounters and relaxation timescales
*timescales for real stellar systems

So far: perfectly smooth potential

In reality: individual stars have a bumpy ride…

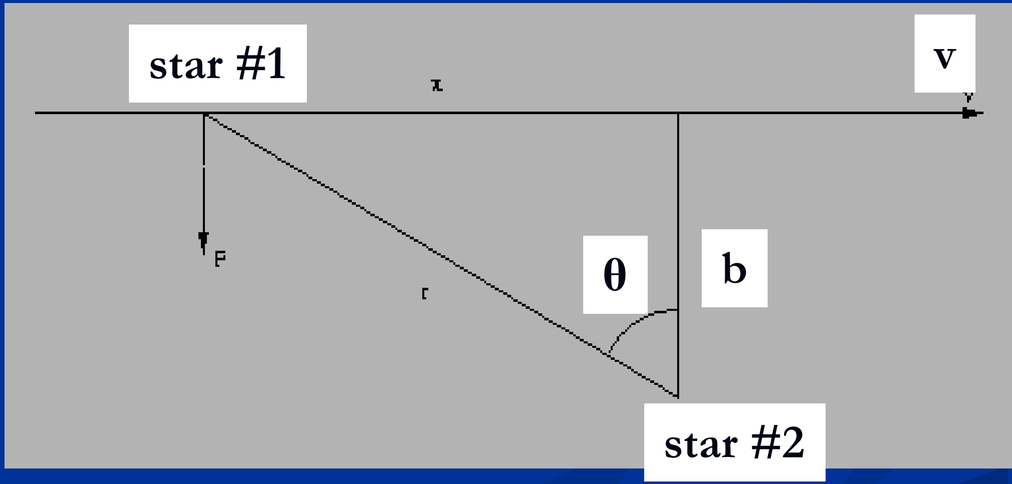
Under these conditions: is the collisionless approximation valid?

- Encounter and (2-body) relaxation timescales

Mean free path (between encounters): $\lambda \sim 1/nA$

where $A \sim b^2$ is cross section (b – impact parameter)

n – number density



Time between encounters:

$\tau \sim 1/nAv$

where v – is the mean velocity between encounters

Consider 3 regimes: direct collision (tidal capture)

strong encounters

weak encounters

❖Direct collision (tidal capture)

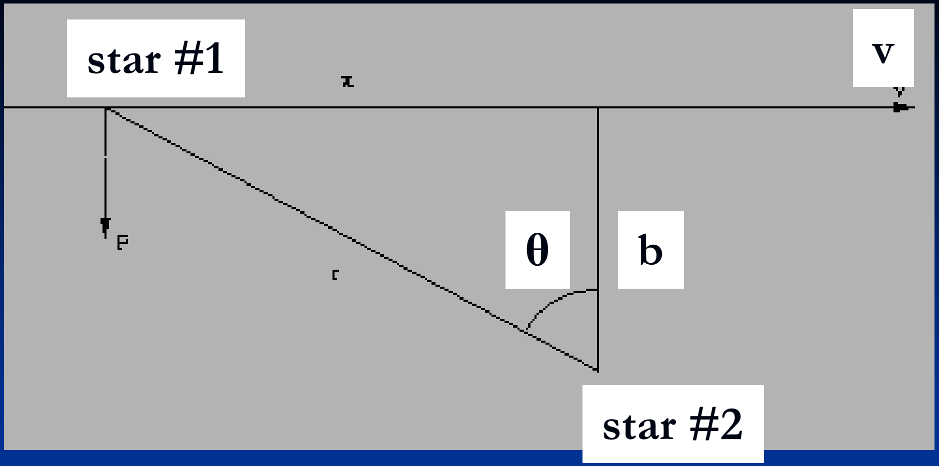
For b~few x R_* (where R_* is the stellar radius):

strong tides → orbital energy dissipates, tidal capture

depending on circumstances, stars can coalesce

exceedingly rare in galaxies!

❖Strong deflections



Defined as $\Delta v \sim v$ and occuring when $b = r_s$ (s for strong) is sufficiently small

From virial theorem: $Gm^2/r_s = mv^2$ so $r_s = Gm/v^2$

(where m – stellar mass)

This is about 1 AU where $v \sim 20\text{-}30$ km/s

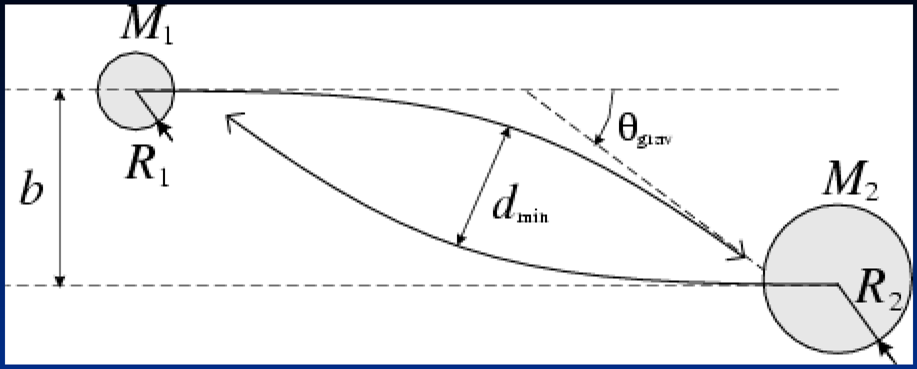
Near Sun: $v \sim 10$ km/s
$n \sim 0.1\ pc^{-3}$

The time interval between collisions, $t_s = 1/nr_s^2 v = v^3/G^2m^2n \sim 10^{15}$ yrs

This is very rare for galaxies, but can be relevant for:
globular clusters cores, galactic nuclei and clusters of galaxies

The Sun hasn't been in a strong encounter since its origin, or the planetary system would be destroyed…

Weak deflections



- Defined as $\Delta \mathbf{v} \ll \mathbf{v}$, so $\mathbf{b} \gg \mathbf{r}_s$

Estimate deflection velocity towards the target star:

Time in the vicinity of a target star is $\Delta t \sim 2b/v$

Perpendicular acceleration is about $\mathbf{Gm/b^2}$ → $\Delta v_\perp \sim 2Gm/bv$

Deflection angle (in the solar neighborhood):

$$\Delta\vartheta \sim \Delta v_\perp / v \sim 2Gm/bv^2 \sim 2''$$

- After many encounters: $\Delta v_{tot} \sim \sum \Delta v$

Since the deflection orientations are random:

Δv_{tot} performs random walk

The amplitude (squared) of this random walk after time t:

$$d^2 \sim \sum \lambda^2 N \sim N \sum (\Delta v_\perp)^2$$

$$|\Delta V_{\rm tot}|^2 = \sum |\Delta V|^2 = \int_{b_{min}}^{b_{max}} \left(\frac{2Gm}{bV}\right)^2 t\, nV\, 2\pi b\, db \qquad \text{(4-1a)}$$

$$= \frac{8\pi G^2 m^2 n t}{V} \ln \Lambda \qquad \text{(4-1b)}$$

where $\Lambda = b_{max}/b_{min}$

Relaxation time: when $\Delta v_{tot} \sim v$. So changing $v \rightarrow \sigma$,

$$t_{\rm relax} \simeq 0.34 \frac{\sigma^3}{G^2 m \rho \ln \Lambda}$$

$$\simeq \frac{1.8 \times 10^{10}\ \rm yr}{\ln \Lambda} \sigma_{10}^3\, m_\odot^{-1}\, \rho_3^{-1}$$

where $[\sigma] = 10$ km/s; $[m] = M_\odot$; $[\rho] = 10^3 M_\odot/pc^3$

•Alternative expressions for (2-body) relaxation time

Using eq.(4-1b)

$$|\Delta V_{\rm tot}|^2 = \sum |\Delta V|^2 = \frac{8\pi G^2 m^2 n t}{V} \ln \Lambda$$

and a system of a size **R** containing **N** stars: $\mathbf{n = 3N/4\pi R^3}$

From the virial theorem: $\mathbf{v^2 = GM/R = GNm/R}$

and the system relaxes after $\mathbf{\Delta v_{tot} = v}$

Take $\mathbf{b_{max} \sim R}$, $\mathbf{b_{min} \sim r_s = Gm/v^2} \rightarrow \mathbf{b_{max}/b_{min} = \Lambda \sim N}$

Next choose units of time as crossing time: $\mathbf{t = t_{cross} \sim R/v}$

Substituting:

$$t_{\rm relax} \simeq t_{\rm cross} \frac{N}{6 \ln N}$$

The answer depends only on **N** (!)

Note, that in $t_{\rm relax} \simeq t_{\rm cross} \dfrac{N}{6 \ln N}$ $\mathbf{t_{relax}} > \mathbf{t_{cross}}$ for **N** > 30

So, to a good approximation: stars orbit in overall potential

Equal logarithmic intervals in **b** have equal contribution to long term deflections : from **R** to **0.5R**, from **0.5R** to **0.25R**

However, because of $\mathbf{\Delta\vartheta \sim \Delta v_{\perp}/v \sim 2Gm/bv^2 \sim 2''}$

$\mathbf{\Delta v/v \sim 1/b}$ → for systems with **R >> b**$_{\mathbf{min}}$**,** most scattering is due to weak encounters (**Δv << v**)

➢Timescales in real stellar systems

System	N	R (pc)	t_{cross}	t_{relax}	t_{age}	age/relax
Open cluster	10^2	2	10^6	10^7	10^8	10
Glob. cluster	10^5	4	$10^{5.6}$	$10^{8.4}$	10^{10}	20
Elliptical g-y	10^{11}	$10^{4.5}$	10^8	$10^{16.4}$	10^{10}	10^{-7}
Gal. nucleus	10^8	10	$10^{4.2}$	10^{10}	10^{10}	1
Cluster of g-s	10^2	$10^{5.6}$	10^9	$10^{9.3}$	10^{10}	3

2-body relaxation may be relevant for galactic nuclei and clusteres of galaxies, but can be completely neglected for galaxies

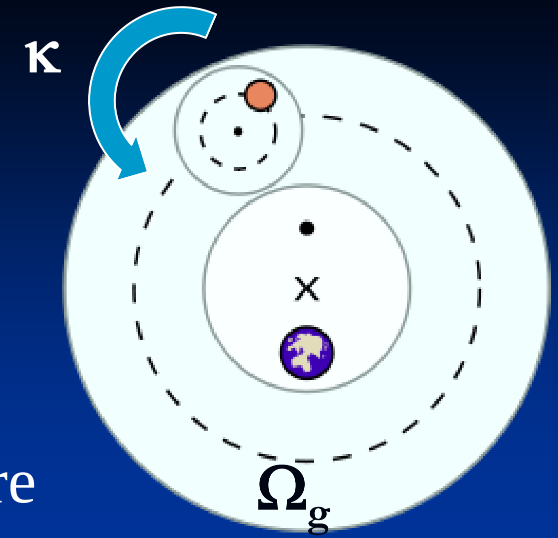
DYNAMICS OF GALACTIC DISKS

Stellar orbits:

*epicyclic approximation
*resonances
*density waves
*disk instabilities

➢Epicyclic approximation

❖Overview



•Assume: disk stars have circular trajectories with small deviations

Use Ptolemaic approximation – stellar orbits are superpositions of:

circular orbits along guiding center (deferent), radius $\mathbf{R_g}$, angular velocity $\mathbf{\Omega_g}$

smaller elliptical epicycle, angular velocity $\boldsymbol{\kappa}$, retrograde

•Consider gravity/centrifugal balance and conservation of angular momentum (AM = J)

put the star at the guiding center (GC) and perturb it radially outward

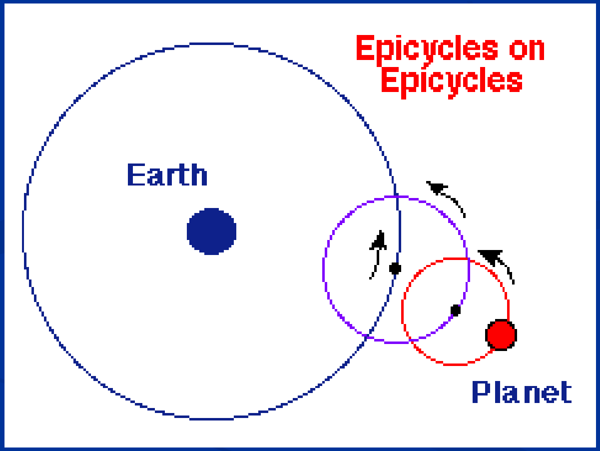
conserving $\mathbf{AM = mrv_\phi}$ → increase in r means the star moves backwards (relatively to GC)

The new balance between gravity and centrifugal forces:

if **AM = const** → $\mathbf{F_{centrifug} \sim v_\phi{}^2/r \sim r^{-3}}$ while $\mathbf{F_{grav}}$ fall slower than $\mathbf{r^{-2}}$

at larger **r**: $\mathbf{F_{grav} > F_{centrifug}}$ and the star is pulled inwards (relative to GC)

as the star falls in, **r** decreases and **v** increases → the star moves forward (rel to GC)



but $\mathbf{F_{centrifug} > F_{grav}}$ and the star moves forward again

the cycle repeats → a star moves on a small retrograde epicycle

•In general, $\mathbf{\Omega_g}$ and $\boldsymbol{\kappa}$ are different → so the orbit does not close

however, as we shall see: from the reference frame rotating with $\mathbf{\Omega_g - \kappa/2}$ → the orbits do nearly close (closed ellipses)

- Consider a smooth axisymmetric flattened mass distribution with potential **$\Phi(R,z)$** and **J = const.** (no azimuthal forces!), and using cylindrical coordinate system:

$$\ddot{\mathbf{R}} = -\nabla\Phi(R,z); \qquad L_z = R^2\dot\varphi = \mathrm{const}$$

Test particle velocity in the z=0 plane: $v^2 = \dot R^2 + (R\dot\phi)^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Lagrangian: $L = K - \Phi$

$$L = 0.5m[\dot R^2 + (R\dot\phi)^2] - m\Phi$$

$$\frac{\partial L}{\partial \dot R} = m\,\dot R \qquad \frac{\partial L}{\partial R} = mR\dot\phi^2 - \frac{\partial \Phi}{\partial R}$$

$$\frac{\partial L}{\partial \dot\phi} = m\,R^2\dot\phi \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot R}\right) - \left(\frac{\partial L}{\partial R}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot \phi}\right) - \left(\frac{\partial L}{\partial \phi}\right) = 0$$

$$\frac{d}{dt}(m\dot R) - mR\dot\phi^2 + \frac{d\Phi}{dR} = 0$$

$$\frac{d}{dt}(mR\dot\phi) = 0$$

$$\frac{d}{dt}\left(m\dot{R}\right) - mR\dot{\phi}^2 + \frac{d\Phi}{dR} = 0$$

$$\frac{d}{dt}\left(mR\dot{\phi}\right) = 0$$

Separating motions along each coordinate:

in the plane z=0

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R}\ ; \qquad \frac{d}{dt}(L_z) = 0\ ; \qquad \ddot{z} = -\frac{\partial \Phi}{\partial z}$$

net acceleration – centrifugal = gravity

out of z=0 plane

Vertical (z) motions

Since the disk is symmetric with respect to **z=0**:

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = 0$$

Consider small motions about the plane **z=0**:

$$\ddot{z} = -\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} - z\left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{z=0} = -z\left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{z=0} = -\nu^2 z$$

This results in a simple harmonic motion with (vertical) frequency ν:

$$z(t) = Z\cos(\nu t + \psi_0)$$

For MW galaxy near the Sun: $\nu^2 = \mathbf{4\pi G\rho_0} \rightarrow \mathbf{2\pi}/\nu \sim 6.5\ 10^7$ yrs

~ 1/3 of the circular period

Radial motions

- Consider first the circular motion with **R=const** (it has **$R=R_g$**, circular velocity **v_c** and **Ω_g**):

$$-\left(\frac{\partial \Phi}{\partial R}\right)_{R_g} = \frac{V_c^2}{R_g} = R_g \Omega_g^2$$

For non-circular orbits, the radial acceleration is given by:

$$\ddot{R} = R\dot{\phi}^2 - \frac{\partial \Phi}{\partial R} \qquad (4\text{-}2)$$

where the 2nd term on the right is the forcing term

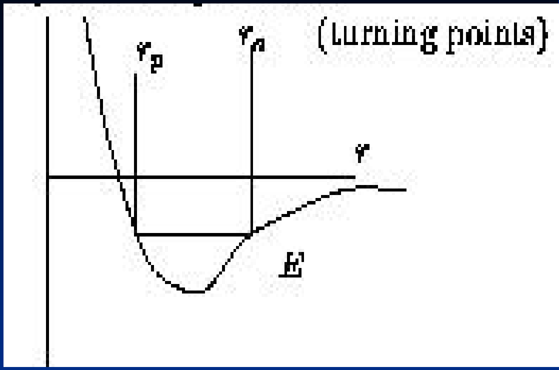
- This can be written as

$$\ddot{R} = \frac{\partial \Phi_{\rm eff}}{\partial R} \quad \text{where} \quad \Phi_{\rm eff} = \Phi(R, z) + \frac{L_z}{2R^2}$$

and **Φ_{eff}** is the effective potential

The effective potential behaves:

at small **R**: sharp increase → centrifugal barrier
at large **R**: slow increase
minimum: at $\mathbf{R=R_g}$ → circular orbit of the GC



$$\left(\frac{\partial \Phi_{\rm eff}}{\partial R}\right)_{R_g} = 0 = \left(\frac{\partial \Phi}{\partial R}\right)_{R_g} - R_g \dot{\phi}_g^2 = \left(\frac{\partial \Phi}{\partial R}\right)_{R_g} - \frac{V_c^2}{R_g}$$

•Other orbits will oscillate around $\mathbf{R_g}$

Consider the potential at $\mathbf{R=R_g+x}$ (where **x** – small perturbation):

$$\ddot{R} = \ddot{x} = -\left(\frac{\partial \Phi_{\rm eff}}{\partial R}\right)_{R_g} - x\left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{R_g} = -x\left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{R_g} = -\kappa^2 x$$

harmonic motion about the GC: $x(t) = X\cos(\kappa t + \phi_0)$

with an epicyclic frequency **κ**, where

$$\kappa^2 = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{R_g} = \left(\frac{\partial}{\partial R}\left(\frac{\partial \phi}{\partial R}\right)\right)_{R_g} + \frac{L_z^2}{R_g^4} = \left(R\frac{d\,\Omega^2}{d\,R} + 4\Omega^2\right)_{R_g}$$

❖ Azimuthal motion

Azimuthal acceleration = 0 → $L_z = R^2 \dot{\varphi} = \text{const}$

In cylindrical coordinates, this means $2\dot{R}\dot{\varphi} + R\ddot{\varphi} = 0$ (4-3)

Since $L_z = R_g^2 \Omega_g = R^2\Omega = \text{const.}$: changes in **R** → changes in **Ω**

Remember: $\Omega = \dot{\varphi}$

$$\dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{(R_g + x)^2} \simeq \frac{L_z}{R_g^2}\left(1 - \frac{2x}{R_g}\right) = \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

Integration results in:

$$\phi(t) = \Omega_g t - \frac{2\Omega_g X}{\kappa R_g} \sin(\kappa t + \phi_0)$$

→ $\phi(t)$ follows the GC with a small amplitude harmonic motion superposed

Taking **y**-axis in the forward direction with the origin on the GC:

$y = R_g\, \phi(t)$ →

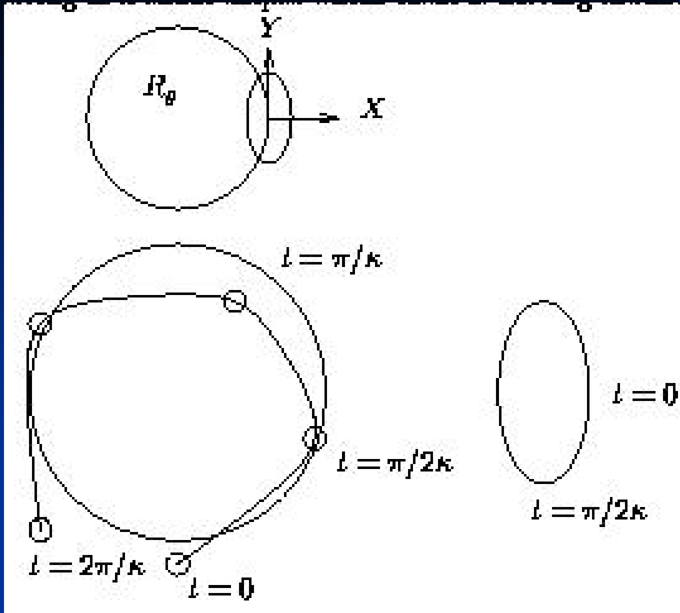
$$y(t) = -\frac{2\Omega_g}{\kappa} X \sin(\kappa t + \phi_0)$$

the same **κ**, but out of phase by 90°

Summarizing and taking $\phi_0 = 0$, we have:

$x = X \cos(\kappa t)$

$y = -(2\Omega/\kappa)\, X \sin(\kappa t)$



elliptical epicycle with radial/azimuthal axis ratio $\kappa/2\Omega$

epicyclic motion is retrograde (Ptolemy's was prograde!)

for stars with the same R_g, the velocity ellipsoid is $\sigma_R/\sigma_\theta \sim \kappa/2\theta$

for Keplerian potential: $\Omega \sim R^{-3/2} \rightarrow \kappa = \Omega$ closed ellipse centered at the ellipse focus with axis ratio 2:1 (Ptolemy's: 1:1 circles)

for flat rotation curve: $\Omega \sim R^{-1} \rightarrow \kappa^2 = 2\Omega^2$

for solid body rotation: $\kappa = 2\Omega$

in general: $\Omega < \kappa < 2\Omega \rightarrow \kappa > \Omega$ and the epicycle is completed before rotation

➢Resonances

- Rotating patterns: observations show that spiral arms and stellar bars are density enhancements. Their patterns are neither stationary or move with the stars.

- Instead, they move with their own angular speed Ω_p called pattern speed

- Interactions of this pattern speed with epicyclic motion

resonances

❖Orbits in a spiral pattern

Describe the spiral pattern as $\Phi + \delta\Phi$:

$$\delta\Phi(R,\varphi,t) = \delta\Phi(R)\, e^{i(m\varphi - \omega t)} \qquad m=0, 1, 2, \ldots \qquad (4\text{-}4)$$

At fixed **R**, a point of a constant phase = $m\phi - \omega t$, so

$$\dot{\varphi} = \frac{\omega}{m} \equiv \Omega_p$$

← pattern frequency

This is how fast the pattern goes around

Using eq.(4-2) for radial oscillations: $\ddot{R} = R\dot{\phi}^2 - \frac{\partial \Phi}{\partial R}$

forcing term

We add perturbation given by eq.(4-4):

radial:

$$\ddot{R} - R\dot{\varphi}^2 = -\frac{\partial\Phi}{\partial R} - \frac{\partial\delta\Phi}{\partial R}\cos(\ldots) \qquad (4\text{-}5a)$$

The same for azimuthal motion given by eq.(4-3): $2\dot{R}\dot{\varphi} + R\ddot{\varphi} = 0$

azimuthal:

$$R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = m\,\delta\Phi(R)\sin(\ldots) \qquad (4\text{-}5b)$$

azimuthal gradient of perturbing potential

To solve eqs.(4-5ab):

$$\ddot{R} - R\dot{\varphi}^2 = -\frac{\partial \Phi}{\partial R} - \frac{\partial \delta\Phi}{\partial R}\cos(\ldots)$$

$$R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = m\,\delta\Phi(R)\sin(\ldots)$$

use

$$R = R_0 + \delta R$$
$$\varphi = \varphi_0 + \delta\varphi$$

and assume δA<<A (amplitudes) ← good except at the resonance

Solution:

$$\delta R = \delta A \frac{1}{\Delta}\cos[m\varphi_0 + m(\Omega_0 - \Omega_p)t]$$

where

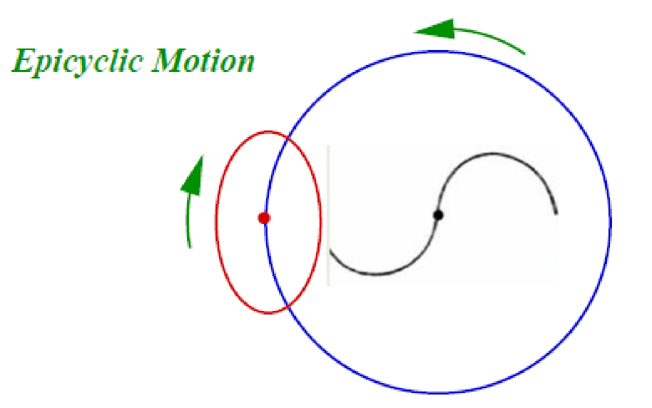
$$\Delta = \kappa_0{}^2 - m^2(\Omega_0 - \Omega_p)^2$$

$$\delta A = -\frac{\partial \delta\Phi}{\partial R} + \frac{2\Omega_0 \delta\Phi(R)}{R_0(\Omega_p - \Omega_0)}$$

Singularities at $\Delta = 0$ → Lindblad resonances
Singularity at $\Omega = \Omega_p$ → corotation resonance

•Corotation resonance: ϕ – component of the force is constant in stars frame

Stars that orbit at the pattern speed $\Omega_* = \Omega_p$ experience a persistent non-axisymmetric perturbation



their response will build up

Can pump energy into either radial or azimuthal part

•Lindblad resonances: consider stars that complete exactly one epicycle between the passage of each arm
→ resonance interaction with each arm

epicyclic amplitude is amplified

where do such resonances occur in a galaxy?

- If the star moves with $\Omega > \Omega_p$ → star overtakes the arm

  If the star moves with $\Omega < \Omega_p$ → arm overtakes the star

$$\delta R = \delta A \frac{1}{\Delta} \cos[m\varphi_0 + m(\Omega_0 - \Omega_p)t]$$

$$\Delta = \kappa_0^2 - m^2(\Omega_0 - \Omega_p)^2$$

$$\delta A = -\frac{\partial \delta\Phi}{\partial R} + \frac{2\Omega_0 \delta\Phi(R)}{R_0(\Omega_p - \Omega_0)}$$

Angular frequency of encountering the arms is $m(\Omega_p - \Omega)$

↓

condition for a resonance:

$$m(\Omega_p - \Omega) = \pm\kappa$$

→

$$\Omega_p - \Omega = \pm\kappa/m$$

Note that $\Omega_p - \Omega = -\kappa/2$ is a special case of a two-armed (m=2) spiral with $\Omega > \Omega_p$

- There are two classes of Lindblad resonances:

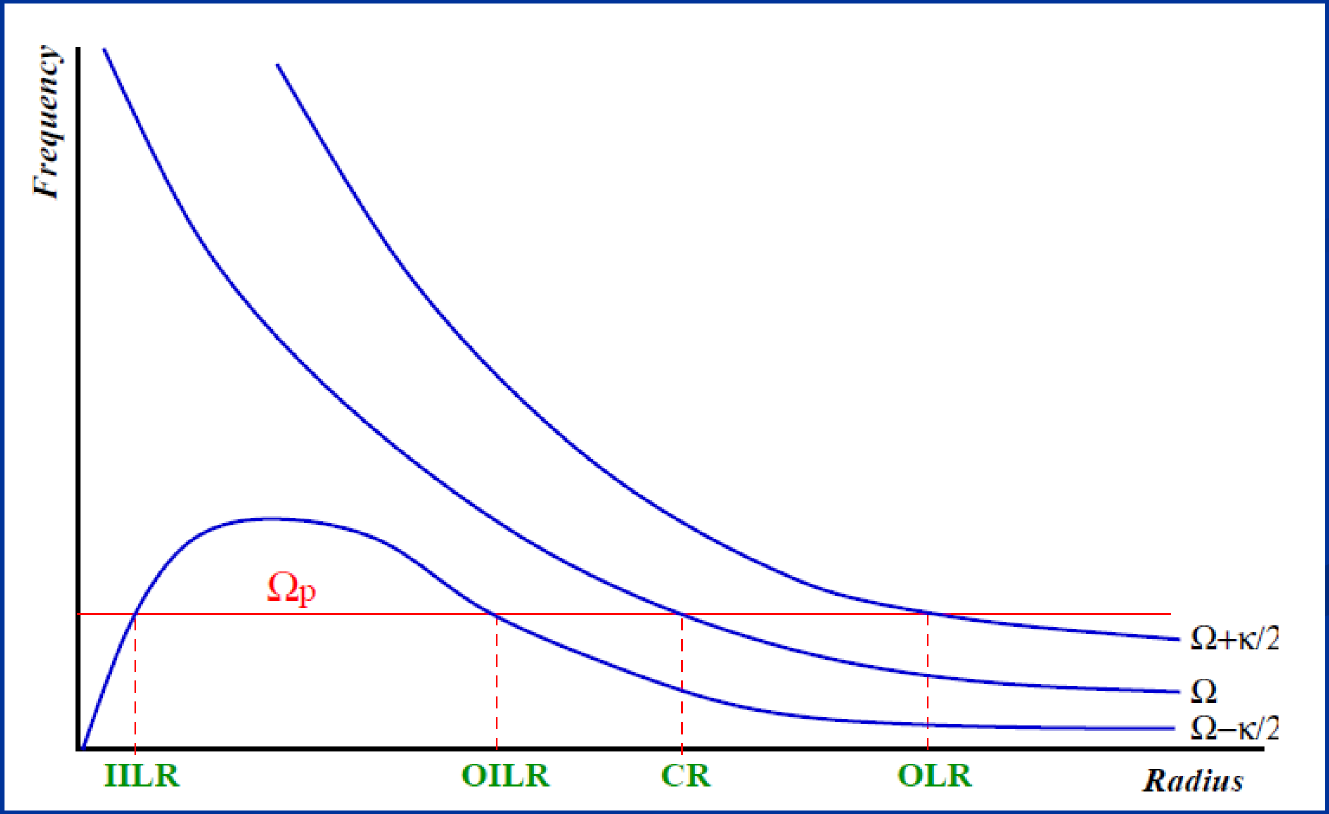
$\Omega_p - \Omega = -\kappa/2$	Inner Lindblad resonance (ILR)
$\Omega_p - \Omega = +\kappa/2$	Outer Lindblad resonance (OLR)

•To establish Lindblad and Corotation resonances: need to know

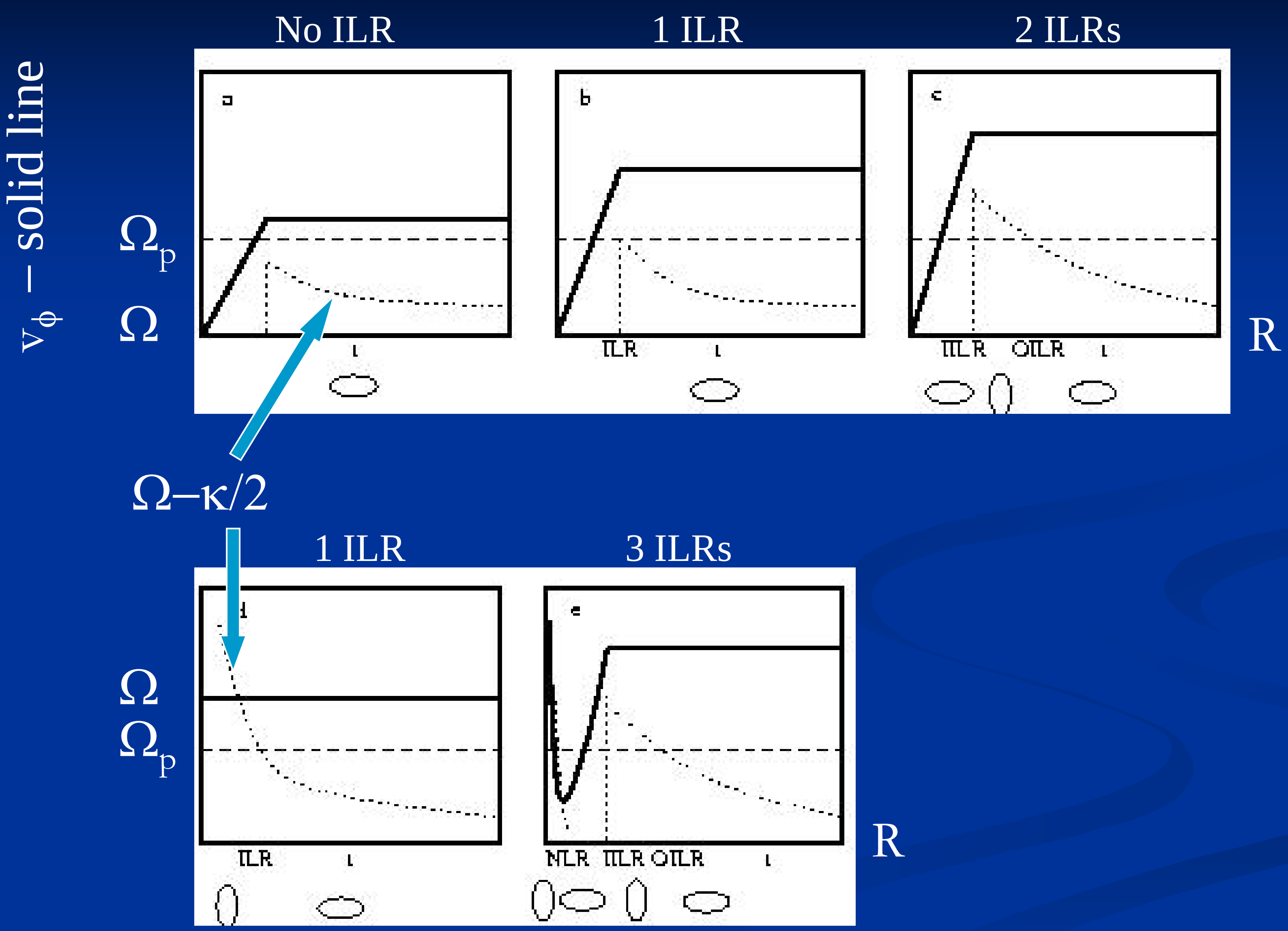
The pattern speed Ω_p

The rotation curve $v_c(R) \rightarrow \Omega(R),\ \kappa(R)$

•For **m=2**: the resonance curves for nearly circular orbits are



•Different types of ILRs for various rotation curves (m=2):



There can be 0, 1, 2 or more ILRs

By superposing many concentric ovals → variety of spiral patterns:

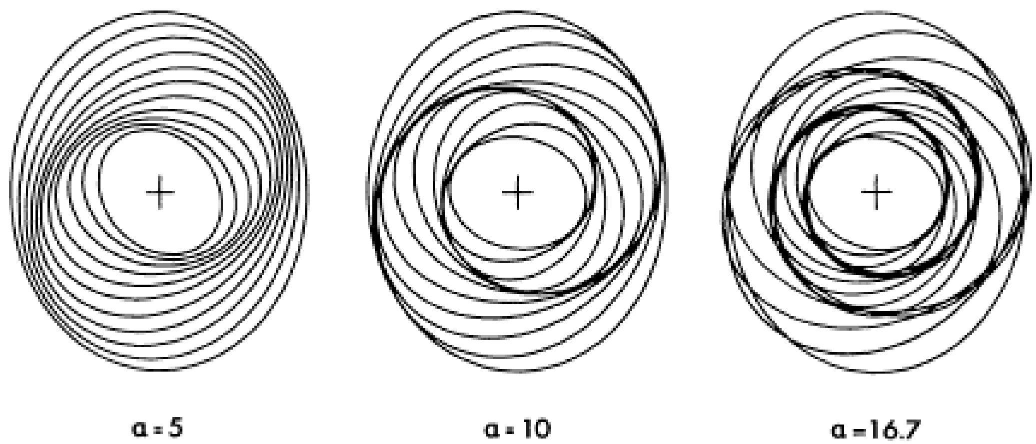
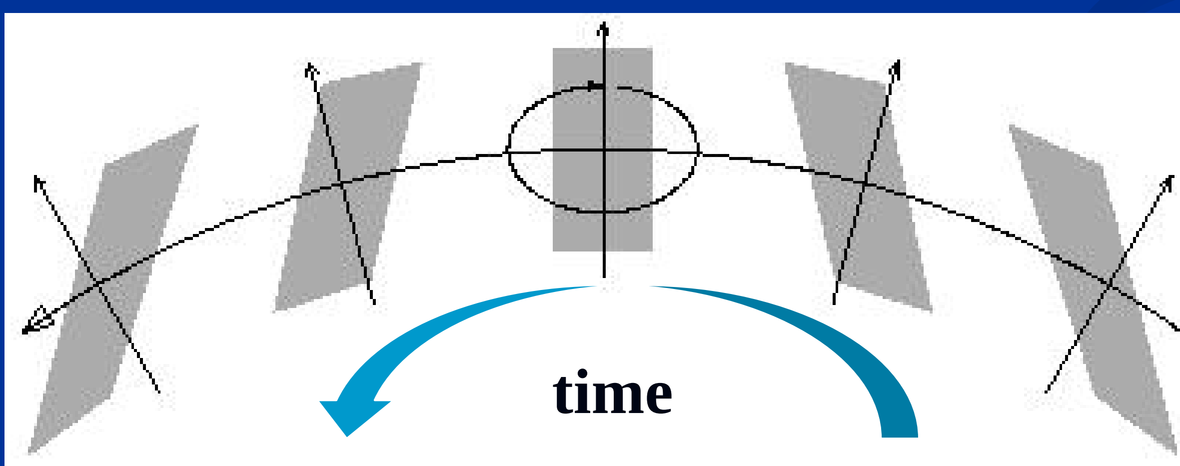


Figure 3 Three kinematic density waves of spiral form (Kalnajs 1973), obtained by merely superposing some kinematic ring waves such as shown in Figure 2.

If $\Omega-\kappa/2$ is independent of radius
→the spiral pattern will persist indefinitely, because all superposed ovals would precess at the same rate

In fact, $\Omega-\kappa/2$ is nearly independent of radius in real disk galaxies over a range in **R**!



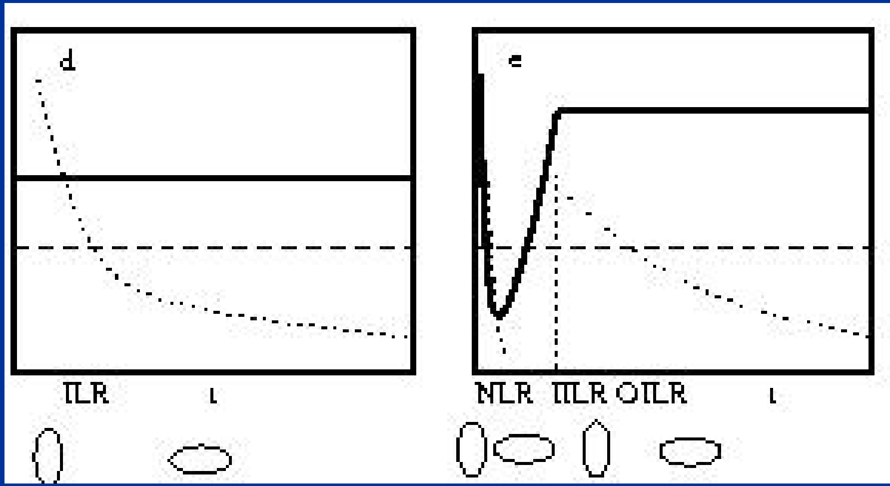
Evolution of an overdense perturbation (gray patch) in a shearing disk which rotates **counter**-clockwise. The perturbation initially has a form of a *leading* spiral, but is sheared into a *trailing* one. The epicycle and the perturbation rotate in the same direction, so stars stay in the perturbation longer.

Density wave spirals should wind up 6 times slower than material arms

❖Importance of resonances

Response changes at resonances: example of 1-D oscillator

Stellar orbits change orientation abruptly at the resonances:



Stellar density waves cannot propagate across the resonances