Dynamical State of the ISM

*Laws of gas dynamics
*Jeans instability in the gas

Laws of gas dynamics

Hydrodynamical regime

- in the ISM: λ << size of the system
  ↑ m.f.p.
- v—distribution → Maxwellian ← T, P, r averaged over many particles (macroscopic variables)
- equations of motion (**r**,t) of gas using macroscopic parameters

❖Conservation principles

- derive equations of motion using conservation principles:

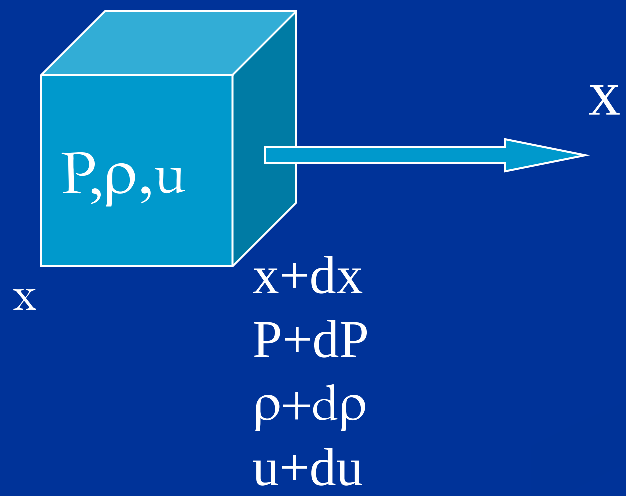
mass, momentum, energy

- conservation principle for quantity Π within volume V:

$$\left\{ \begin{array}{c} \text{rate of increase of } \Pi \\ \text{in V} \end{array} \right\} = \left\{ \begin{array}{c} \text{net convection} \\ \text{into V} \end{array} \right\} + \left\{ \begin{array}{c} \text{net generation} \\ \text{rate in V} \end{array} \right\}$$

- assume plane-parallel geometry: all variables are functions of x:

$dV \rightarrow dx$



• mass conservation

$$\mathbf{\Pi \equiv \rho\, dx}$$

If no sources or sinks (in V):

$$\frac{\partial}{\partial \mathbf{t}}(\boldsymbol{\rho}\,\mathbf{dx}) = \boldsymbol{\rho}\mathbf{u} - (\boldsymbol{\rho} + \mathbf{d}\boldsymbol{\rho})(\mathbf{u} + \mathbf{du})$$

Leaving 1st order terms only:

$$\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{x}} + \boldsymbol{\rho}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{0}$$

eq. of continuity (5-1a)

This is equation of mass conservation at a particular point **r** (Eulerian system of coordinates)

This equation can be written in Lagrangian form →

In Eulerian form:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

In Lagrangian form:

$$\frac{d}{dt} \quad or \quad \rightarrow \frac{D\rho}{Dt} = -\rho\frac{\partial u}{\partial x}$$

$$\equiv \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}$$

convective derivative (carried with fluid element)

advection term

In 3-D:

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot v \quad \Rightarrow \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j} = \frac{\partial}{\partial t} + (v\cdot\vec{\nabla})$$

- Conservation of motion

$$\Pi \equiv \rho u \, dx$$

Assume: gas pressure is the only force acting on the gas

$$\frac{\partial}{\partial t}(\rho u \, dx) = \rho u^2 - (\rho + d\rho)(u + du)^2 + P - (P + dP)$$

momentum enters

momentum leaves

net change of momentum due to applied force

+continuity eq., leaving 1st order terms only

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

also called Euler eq. (5-1b)

Other forces: gravity, magnetic force, radiation, viscous forces

In 3-D:

$$\rho \frac{D\mathbf{v}}{Dt} \equiv \rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\vec{\nabla})\mathbf{v}\right] = -\vec{\nabla}\mathbf{P} \quad + \vec{\mathbf{f}}$$

$$or \ \frac{\mathbf{d}}{\mathbf{dt}} \qquad \mathbf{v}_j \frac{\partial \mathbf{v}_i}{\partial \mathbf{v}_j}$$

if additional force, say for gravity:

$$(-\rho\,\vec{\nabla}\mathbf{P})$$

where $(\vec{\nabla}\cdot\mathbf{v} = \mathbf{0} \rightarrow \textit{incompressibility})$

•Energy conservation

Adiabatic flow: volume element neither gains nor loses heat (by contact with surroundings, or by radiation, etc.)

↓

Change of internal energy = work on surroundings

↓

Entropy=const → $\mathbf{P=K\rho^{\gamma}}$ where γ is ratio of specific heats

If **K=const** → isentropic flow **(γ=5/3)**

Isothermal flow:

$$\mathbf{P = \frac{\rho k T}{\mu m}}$$

- Sound waves

Assume that gas which is uniform and at rest initially is perturbed:

$$\mathbf{P} = \mathbf{P_0} + \mathbf{P_1}$$
$$\boldsymbol{\rho} = \boldsymbol{\rho_0} + \boldsymbol{\rho_1}$$
$$\mathbf{u} = \mathbf{u_1}$$

$\mathbf{u_0}$ and gradient of ρ_0 (unperturbed) are equal zero!

$$\mathbf{P} = \mathbf{K}\boldsymbol{\rho}^{\mathbf{n}} \rightarrow \mathbf{P_1} = \mathbf{nK}\boldsymbol{\rho_0}^{\mathbf{n-1}} \equiv \mathbf{n}\frac{\mathbf{P_0}}{\boldsymbol{\rho_0}}\boldsymbol{\rho_1}$$

$$\equiv \mathbf{a_0}^2 \quad [cm\ s^{-1}]^2$$

Linear continuity eq.:

$$\frac{1}{\rho_0}\frac{\partial \rho_1}{\partial t} + \frac{\partial u_1}{\partial x} = 0 \qquad (5\text{-}2a)$$

Linear momentum eq.: (Euler)

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} \qquad (5\text{-}2b) \quad \Rightarrow \quad \frac{\partial u_1}{\partial t} = -\frac{a_0^2}{\rho_0}\frac{\partial \rho_1}{\partial x}$$

Then,

differentiating $\frac{1}{\rho_0}\frac{\partial \rho_1}{\partial t}+\frac{\partial u_1}{\partial x}=0$ and $\frac{\partial u_1}{\partial t}=-\frac{a_0{}^2}{\rho_0}\frac{\partial \rho_1}{\partial x}$

(continuity eq.) (Euler eq.)

by $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial x}$

$$\frac{\partial^2 \rho_1}{\partial t^2}-a_0{}^2\frac{\partial^2 \rho_1}{\partial x^2}=0$$

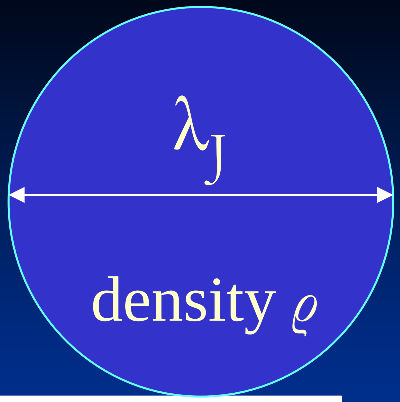
wave equation (5-3)

→ its solution: wave velocity = a_0 → 2 solutions: waves going to $\pm\infty$

$$a_0=\left(\frac{\partial P}{\partial \rho}\right)_s^{1/2}$$

➢ Gravitational Jeans (in)stability

bubble:



without pressure, collapse occurs in a free-fall time

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} = -\frac{4\pi G\rho}{3} r$$

a harmonic oscillator with frequency $\omega = \sqrt{4\pi G\rho/3}$

$$t_{ff} = \frac{1}{4}\frac{2\pi}{\omega} = \left(\frac{\pi^2}{4}\frac{3}{4\pi G\rho}\right)^{1/2} = \left(\frac{3}{16}\frac{\pi}{G\rho}\right)^{1/2}$$

Free-fall timescale:

$$t_{ff} \cong \left(10^6\,\mathrm{yr}\right)\left(\frac{n}{10^3\,\mathrm{cm}^3}\right)^{-1/2}$$

➢Jeans instability in the gas

- The hydro equations:

continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\, \mathbf{v}) = 0$$

Euler:

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{f} - \vec{\nabla} P$$

$$\textit{where} \quad \mathbf{f} = -\rho \vec{\nabla} \Phi$$

Poisson:

$$\Delta \Phi = 4\pi G \rho$$

(5-4a,b,c)

- Linearize all the hydro equations, using

$$P = P_0 + P_1,\ \ \rho = \rho_0 + \rho_1,\ \ \mathbf{v} = \mathbf{v}_1$$

$$\textit{where} \quad \rho_1/\rho_0 << 1,\ \ \textit{etc.}$$

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \vec{\nabla} \mathbf{v}_1 + \mathbf{v}_1 \cdot \vec{\nabla} \cdot \mathbf{v}_0 = -\vec{\nabla}(\Phi_1 + P_1/\rho_0)$$

$$\Delta \Phi_1 = 4\pi G \rho_1$$

(5-5a,b,c)

Assume Jeans initial conditions (homogeneous ISM, no motion)

$$\rho_0 = \text{const}$$
$$v_1 = 0$$

Take $\frac{\partial}{\partial t}$ of

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot v_1 = 0 \qquad (5\text{-}6a)$$

Take divergence of

$$\frac{\partial v_1}{\partial t} = -\vec{\nabla}(\Phi_1 + P_1/\rho_0) \qquad (5\text{-}6b)$$

Use adiabatic motion $P_1 = c_s^2 \rho_1$

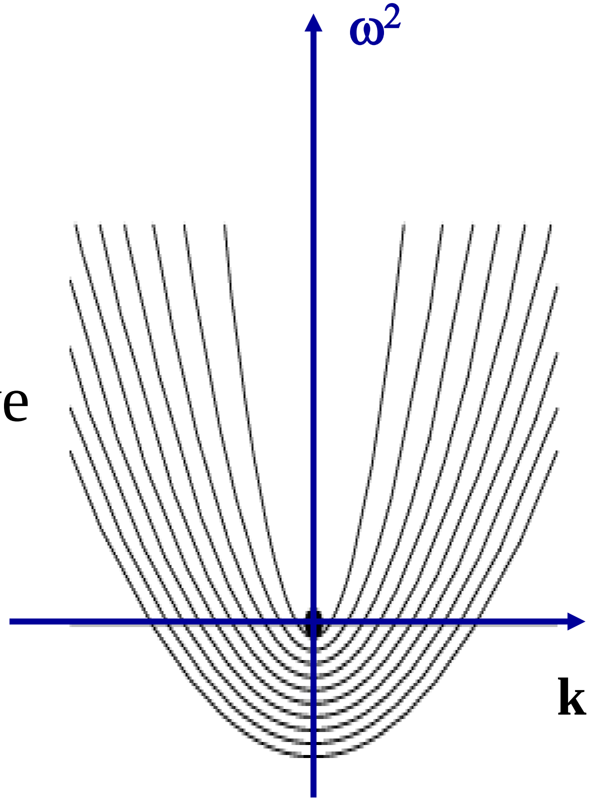
$$\begin{cases} -\dfrac{1}{\rho_0}\dfrac{\partial^2 \rho_1}{\partial t^2} = \dfrac{\partial}{\partial t}(\vec{\nabla} \cdot v_1) \\ \vec{\nabla} \cdot \dfrac{\partial v_1}{\partial t} = \dfrac{\partial}{\partial t}(\vec{\nabla} \cdot v_1) = -\Delta\Phi_1 - (c_s^2/\rho_0)\Delta\rho_1 \end{cases}$$

Eliminate $\Delta\Phi$ and $\frac{\partial}{\partial t}(\vec{\nabla} \cdot v_1)$ from eqs. →

This leads to a wave eq.:

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^{\,2}\Delta\rho_1 - 4\pi G\rho_0\rho_1 = 0$$

Its solution: $\rho_1(x,t) = C(k)e^{i(k\cdot x - \omega t)}$ plane wave



Insert the solution into wave equation:

$$\omega^2 = c_s^{\,2}k^2 - 4\pi G\rho_0 \qquad (5\text{--}7)$$

If $\omega^2 < 0$ → the solution is unstable (grows exponentially!)

Define: $k^2 < k_J^{\,2} \equiv 4\pi G\rho_0/c_s^{\,2}$ (5—8)

where k_J is Jeans wavenumber and $\lambda_J = 2\pi/k_J$

So perturbation is unstable, if $\omega/c_s^{\,2} = k^2 - k_J^{\,2}$

$$\lambda^2 > \lambda_J^{\,2} = \pi c_s^{\,2}/G\rho_0$$

Jeans mass: $$\mathbf{M_J = \frac{4\pi}{3}\rho_0\left(\frac{1}{2}\lambda_J\right)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi c_s^{\ 2}}{G\rho_0}\right)^{3/2}}$$

Typical Jeans mass in the cold molecular phase of the ISM:

$$M_J = \left(\frac{4\pi}{3}\right)\rho R_J^3 = \left(\frac{\pi}{6}\right)\frac{c_s^3}{G^{3/2}\rho^{1/2}} \simeq (2\ \mathrm{M}_\odot)\left(\frac{c_s}{0.2\ \mathrm{km\ s^{-1}}}\right)^3\left(\frac{n}{10^3\ \mathrm{cm^{-3}}}\right)^{-1/2}$$

Analysis of Jeans instability:

The obtained dispersion relation $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$

can be discussed in the limit of no gravity: G → 0

$$\omega^2 = c_s^2 k^2$$

→ sound waves

while in the "cold limit" of $c_s \to 0$:

$$\omega^2 = -4\pi G \rho_0$$

free-fall condition (always unstable)

In a general case: the frequency ω of perturbation is determined by a balance between stabilizing pressure and destabilizing gravity terms