SELF-GRAVITATING DISKS: LOCAL INSTABILITIES

*local stability of gaseous disks
*local stability of stellar disks
*short and long waves

➢Local instabilities in gaseous disks

❖Uniformly rotating infinite sheet (around z-axis with $\mathbf{\Omega} = \Omega e_z$)

continuity: $$\frac{\partial \Sigma}{\partial t} + \vec{\nabla} \cdot (\Sigma \mathbf{v}) = 0$$

Euler: $$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{v} = -\frac{\vec{\nabla} P}{\Sigma} - \vec{\nabla}\Phi - 2\vec{\Omega} \times \mathbf{v} + \Omega^2 (x\hat{e}_x + y\hat{e}_y)$$

Poisson: $$\Delta\Phi = 4\pi G \Sigma \delta(z)$$

(6-1a,b,c)

Assume: the equation of state (barotropic): $P(x,y,t) = P[\Sigma(x,y,t)]$

• In the unperturbed state:

$$\Sigma = \Sigma_0$$
$$\mathbf{v} = 0$$
$$P = P_0 = P(\Sigma_0)$$

Also, in the unperturbed state, all the equations are satisfied trivially

For example, the Euler and Poisson eqs.:

$$\vec{\nabla}\Phi_0 = \Omega^2 (x\hat{e}_x + y\hat{e}_y)$$

$$\Delta\Phi_0 = 4\pi G \Sigma_0 \delta(z)$$

⇒ actually: $\Phi_0 = 2\pi G \Sigma_0 |z|$

Prove!

- In the perturbed state (small perturbations):

$$\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \vec{\nabla} \cdot \mathbf{v}_1 = 0$$

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{c_s^{\,2}}{\Sigma_0} \vec{\nabla}\Sigma_1 - \vec{\nabla}\Phi_1 - 2\vec{\Omega} \times \mathbf{v}_1$$

where $c_s^{\,2} = [dP(\Sigma)/d\Sigma]_{\Sigma_0}$

$$\Delta\Phi_1 = 4\pi G \Sigma_1 \delta(z)$$

(6-2a,b,c)

Eqs.(6-2a,b,c) are similar to (Jeans instability) eqs.(5-6a,b), with addition of Coriolis term

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Euler equation:

for infinite uniformly rotating sheet:

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{c_s^2}{\Sigma_0}\vec{\nabla}\Sigma_1 - \vec{\nabla}\Phi_1 - 2\vec{\Omega}\,x\,\mathbf{v}_1$$

for Jeans instability:

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\vec{\nabla}(\Phi_1 + P_1/\rho_0)$$

Solutions, choose:

$$\boldsymbol{\Sigma}_1(\mathbf{x},\mathbf{y},\mathbf{t}) = \boldsymbol{\Sigma}_a \mathbf{e}^{\mathbf{i}(\mathbf{k}\cdot\mathbf{x}-\omega \mathbf{t})}$$

$$\mathbf{v}_1(\mathbf{x},\mathbf{y},\mathbf{t}) = (\mathbf{v}_{ax}\widehat{\mathbf{e}}_x + \mathbf{v}_{ay}\widehat{\mathbf{e}}_y)\ \mathbf{e}^{\mathbf{i}(\mathbf{k}\cdot\mathbf{x}-\omega \mathbf{t})}$$

$$\boldsymbol{\Phi}_1(\mathbf{x},\mathbf{y},\mathbf{z}=0,\mathbf{t}) = \boldsymbol{\Phi}_a \mathbf{e}^{\mathbf{i}(\mathbf{k}\cdot\mathbf{x}-\omega \mathbf{t})} \qquad \text{(6-3a,b,c)}$$

Also, choose

$$x - axis \,\|\, \mathbf{k},\ so\ \mathbf{k} = k\widehat{\mathbf{e}}_x$$

For Poisson eq.:

$$\Delta\boldsymbol{\Phi}_1 = \mathbf{0} \quad for \quad z \neq 0$$

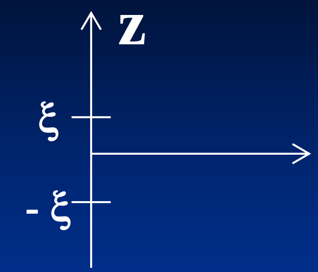
$$\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_a \mathbf{e}^{\mathbf{i}(\mathbf{k}\cdot\mathbf{x}-\omega \mathbf{t})} \quad for \ \ z = 0$$

The only continuous function that satisfies this:

$$\boldsymbol{\Phi}_1(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = \boldsymbol{\Phi}_a \mathbf{e}^{\mathbf{i}(\mathbf{k}\mathbf{x}-\omega \mathbf{t})-|\mathbf{k}\mathbf{z}|}$$

To relate Φ_a to Σ_a, we integrate $\Delta\Phi_1 = 4\pi G\Sigma_1\delta(z)$ across z=0:

from z = - ξ to z = + ξ



$\dfrac{\partial^2\Phi_1}{\partial x^2}$ *and* $\dfrac{\partial^2\Phi_1}{\partial y^2}$ → continuous at z = 0

$\dfrac{\partial^2\Phi_1}{\partial z^2}$ → discontinuous at z = 0

$$\lim_{\xi\to 0}\int_{-\xi}^{+\xi}\frac{\partial^2\Phi_1}{\partial z^2}dz = \lim_{\xi\to 0}\frac{\partial\Phi_1}{\partial z}\Big|_{-\xi}^{\xi} = 4\pi G\Sigma_1\int_{-\xi}^{+\xi}\delta(z)\,dz = 4\pi G\Sigma_1$$

$\Phi_a e^{i(kx-\omega t)-|kz|}$ → $-2|k|\Phi_a = 4\pi G\Sigma_a$

$$\Phi_1(x,y,z,t) = -\frac{2\pi G\Sigma_a}{|k|}e^{i(kx-\omega t)-|kz|}$$

Substituting eqs.(6-3a,b,c) into eqs.(6-2a,b,c):

$$-i\omega\Sigma_a = -ik\Sigma_0 v_{ax}$$

$$-i\omega v_{ax} = -\frac{c_s^{\,2} ik\Sigma_a}{\Sigma_0} + \frac{2\pi G i\Sigma_a k}{|k|} + 2\Omega v_{ay}$$

$$-i\omega v_{ay} = -2\Omega v_{ax}$$

Has nontrivial solution only when

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 |k| + k^2 c_s^{\,2}$$

Dispersion relation for uniformly rotating sheet

Analysis: $\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 |k| + k^2 c_s^2$

The solution $\Sigma_1(x,y,t) = \Sigma_a e^{i(k\cdot x - \omega t)}$

- The sheet is stable if $\omega^2 \geq 0$
- Unstable if $\omega^2 < 0$
- If no rotation: $\Omega = 0$

unstable if $-2\pi G\Sigma_0 |k| + k^2 c_s^2 < 0$

$|k| < k_J = 2\pi G\Sigma_0 / c_s^2$

$k^2 < k_J^2 \equiv 4\pi G\rho_0 / c_s^2$

as eq.(5-8)
for a 3-D case!

Long wavelengths are gravitationally unstable

But there are some differences:

3-D uniform case:

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0$$

nonrotating uniform sheet:

$$\omega^2 = -2\pi G\Sigma_0 |k| + k^2 c_s^2$$

If $c_s \rightarrow 0$

→ $\omega^2 = -4\pi G\rho_0$ for the 3-D uniform case
→ it canNOT be stabilized by internal pressure term! → exponential instability

→ $\omega = \omega(k)$ for the nonrotating uniform sheet
→ it behaves differently and is more violently unstable

Even rotation will not be able to stabilize the sheet at small radii for large enough **k** (small wavelength) without pressure term:

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 |k|$$

Typically, the sheet can be stabilized by a combination of internal pressure and rotation

How can it be stabilized?

The most unstable wavenumber is

$$|k| = \pi G \Sigma_0 / c_s^2 = \frac{1}{2} k_J$$

The sheet is stable at all wavelengths if the minimum $\omega > 0$:

$$\frac{c_s \Omega}{G \Sigma_0} \geq \frac{\pi}{2} = 1.5708$$

For stellar "fluid" → a very similar relation:

$$\frac{\sigma \Omega}{G \Sigma_0} \geq 1.68$$

where σ – stellar dispersion velocity

Uniformly rotating sheet:

- cold sheet is violently unstable
- sheet can be completely stabilized by internal pressure and rotation
- stellar and gaseous sheets behave similarly

❖Differentially rotating gaseous disk: perturbation $\sim e^{i(k \cdot r - \omega t)}$?

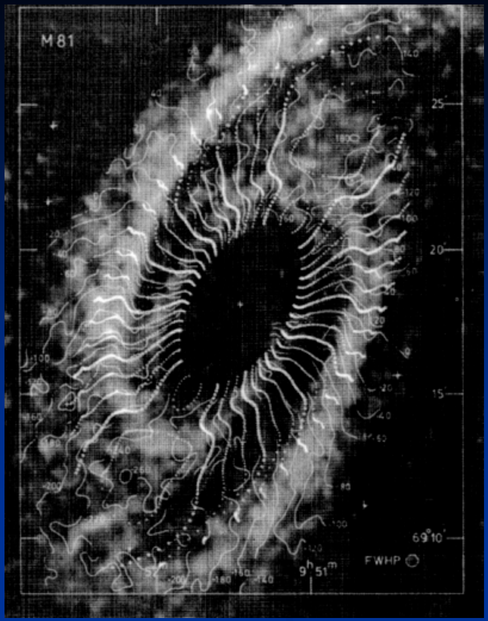
•Gravity is a long range force → we need local perturbation

→ WKB approximation

approximation: tightly-wound spiral  versus 

$$\lambda = \Delta R << R$$

•Response of the disk to a tightly wound spiral arm (WKB approximation):



$$\Sigma_1(x,y,t) = \Sigma_a e^{i(k\cdot x - \omega t)} \Rightarrow \Sigma_1(R,\varphi,t) = \Sigma_a e^{ik(R_0,t)(R-R_0)}$$

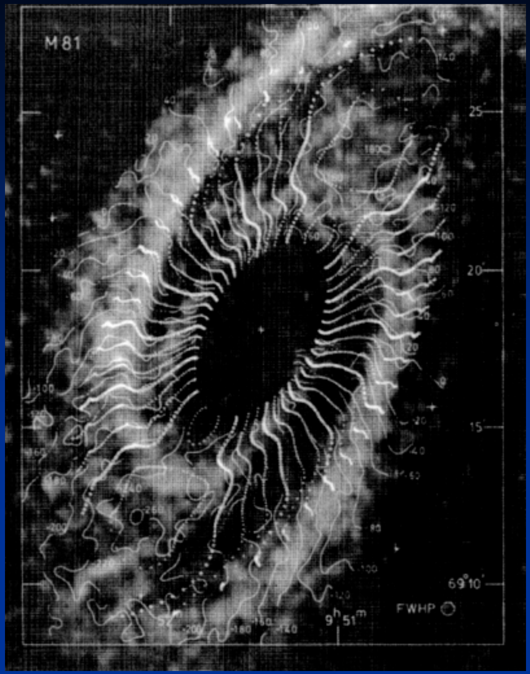
where $$\Sigma_a = A(R_0,t)\,e^{i[m\varphi_0 + mg(R_0,t)]}$$

In the vicinity of (R_0, φ_0) → the spiral arm resembles plane wave

Perturbation: tightly wound spiral

$\Sigma_0(R)$ -- surface density of unperturbed axisymmetric disk

$\Sigma_1(R, \varphi, t)$ -- perturbed surface density



$$\Sigma_1(x, y, t) = \Sigma_a e^{i(k\cdot x - \omega t)}$$ -- perturbation

$$\Sigma_1(R, \varphi, t) = \Sigma_a(R, t)\, e^{i(m\varphi - \omega t)}$$

slow change moving along the arm

rapid change passing between the arms

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 |k| + k^2 c_s^2$$

dispersion relation for gaseous sheet rotating uniformly

- Differentially rotating gaseous disk: tightly-wound perturbation
  - Using the same perturbation technique, but now **Ω=Ω(r)** and:

$$\kappa^2(R) = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

where **κ** -- is the epicyclic frequency

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^2$$

dispersion relation for gaseous disk in a tightly-wound limit

where we applied *m*-armed perturbation (spiral)

<u>3 terms:</u> angular momentum (stabilizes at large spatial scales)

gravity (destabilizes intermediate scales)

pressure (stabilizes at small spatial scales)

  - For uniform rotation: $\kappa = 2\Omega$
- Differentially rotating stellar disk:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma |k| F\left(\frac{\omega - m\Omega}{\kappa}; \frac{k^2\sigma_r^2}{\kappa^2}\right)$$

where **F(…)** is the reduction factor

❖Local axisymmetric stability of differentially rotating disks

- Gaseous disks:

Axisymmetric perturbation → **m = 0**:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^{\,2}$$

If $\omega^2 < 0 \;\rightarrow\; \omega = \omega_{real} + i\omega_{im}$ and $e^{-i\omega t} = e^{-i\omega_{real} t}\, e^{+\omega_{im} t}$

Neutral stability:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^{\,2} = 0$$

with $\lambda_{crit} \equiv \dfrac{4\pi^2 G\Sigma}{\kappa^2}$

Stable if $\omega^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^2 \geq 0$

Solve the quadratic equation for |k| and introduce the Toomre parameter (Q):

Stable if $c_s^2 k^2 - 2\pi G\Sigma |k| + \kappa^2 \geq 0$

$$k^2 - (2\pi G\Sigma / c_s^2)|k| + (\kappa / c_s)^2 \geq 0$$

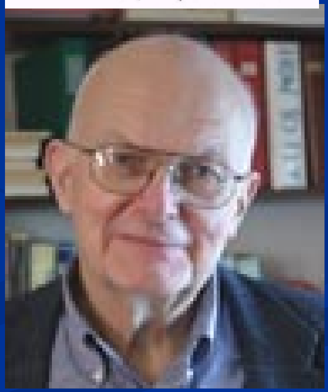
$$k^2 - 2(\pi G\Sigma / c_s^2)|k| + (\pi G\Sigma / c_s^2)^2 - (\pi G\Sigma / c_s^2)^2 + (\kappa / c_s)^2 \geq 0$$

$$[k - (\pi G\Sigma / c_s^2)]^2 - (\pi G\Sigma / c_s^2)^2 + (\kappa / c_s)^2 \geq 0$$

stable, if $-(\pi G\Sigma / c_s^2)^2 + (\kappa / c_s)^2 \geq 0$

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma} > 1 \quad (\textit{for gas})$$

Toomre's parameter



only local stability!

Unstable waves imply star or galaxy formation

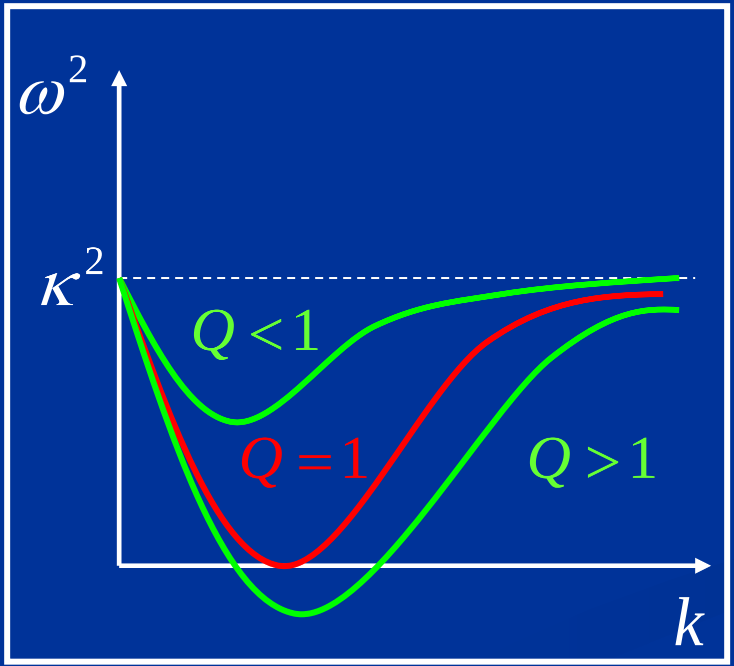
•For stellar disk, the local axisymmetric stability:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k| F(\alpha, k^2\sigma_r^2/\kappa^2) \geq 0$$

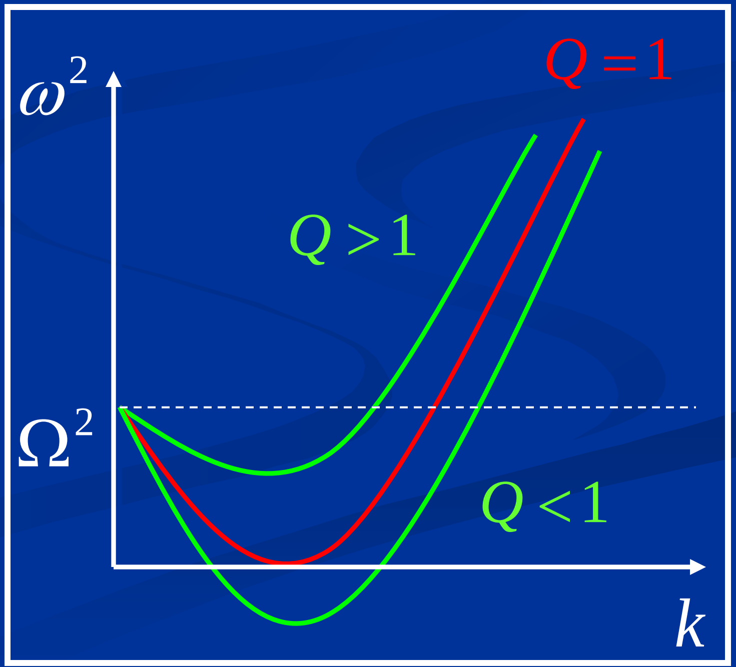
where $\alpha = (\omega - m\Omega)/\kappa$

$$Q \equiv \frac{\sigma_r \kappa}{3.36\, G\Sigma} > 1 \quad (\textit{for stars})$$

stellar disks



gaseous disks



• Neutral stability for m=0: $Q(\lambda / \lambda_{crit})$

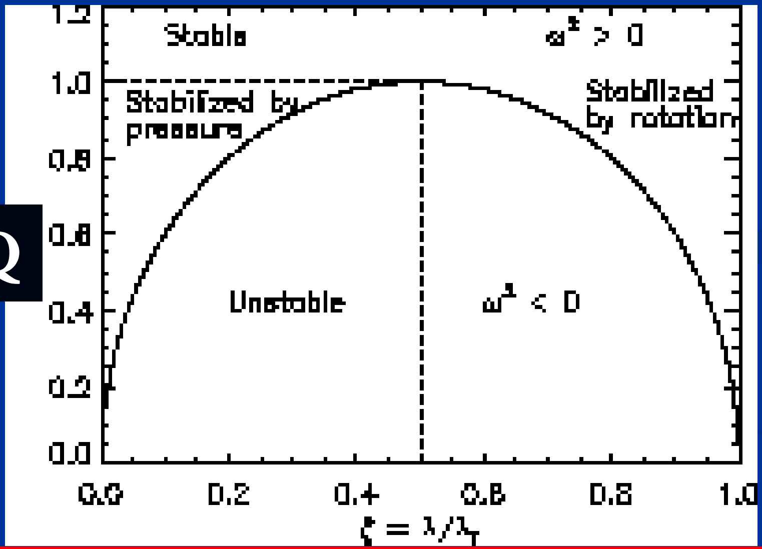
$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^2 \geq 0$$

gaseous disk

$$\omega^2 = \kappa^2 - 2\pi G\Sigma |k| F(\alpha, k^2\sigma_r^2/\kappa^2) \geq 0$$

stellar disk

m = 0 → Q



$\lambda(\text{most unstable}) = p\lambda_{crit}$

→ most unstable wavelength

p=0.5--0.55 for h=0

Neutral stability curve for small λ axisymmetric perturbations in gaseous disk. Solid line – marginal stability. Short wavelength stabilizes by pressure, long – by rotation. Above **Q=1** – no axisymmetric instability can grow!

- Nonaxisymmetric perturbations: waves in differentially rotating gaseous and stellar disks
  - Gaseous disks: $(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^2$

$$k^2 c_s^2 - 2\pi G\Sigma |k| + \kappa^2 - m^2(\Omega - \Omega_p)^2 = 0 \quad \leftarrow \quad \Omega_p = \omega/m$$

Solving this for |k|:

$$|k| = \frac{2\pi G\Sigma \pm \sqrt{4\pi^2 G^2 \Sigma^2 - 4c_s^2[\kappa^2 - m^2(\Omega - \Omega_p)^2]}}{2c_s^2}$$

or multiplying by $\kappa/\pi G\Sigma$

$$|k| = \frac{\kappa \pm \sqrt{\kappa^2 - Q^2\kappa^2 + m^2 Q^2 (\Omega - \Omega_p)^2]}}{c_s Q}$$

If $\sqrt{\ldots} < 0$ → $\kappa^2 - Q^2\kappa^2 + m^2 Q^2 (\Omega - \Omega_p)^2 < 0$ wave damping

Again

$$|\mathbf{k}| = \frac{\kappa \pm \sqrt{\kappa^2 - Q^2\kappa^2 + m^2Q^2(\Omega - \Omega_p)^2]}}{c_s Q}$$

This means, if the expression under the sign of $\sqrt{\ldots} < 0$

$$\kappa^2 - Q^2\kappa^2 + m^2Q^2(\Omega - \Omega_p)^2 < 0$$

|k| is complex number → wave damping

If Q =1 → $\kappa^2 - \kappa^2 + m^2(\Omega - \Omega_p)^2 \quad \text{never} \quad < 0$

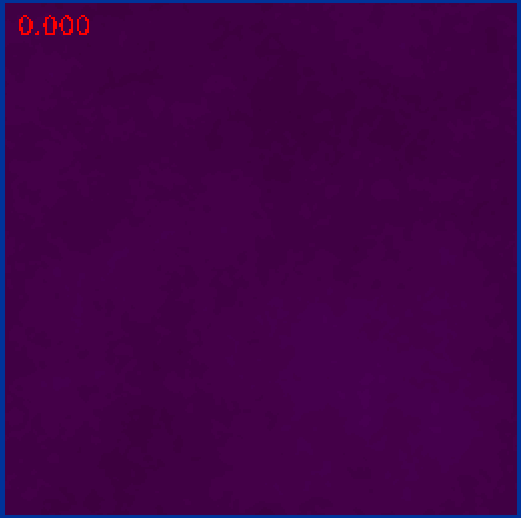
If Q > 1 → forbidden zone around corotation: $\Omega = \Omega_p$ (see later!)

$$\kappa^2 - Q^2\kappa^2 + m^2Q^2(\Omega - \Omega_p)^2 < 0$$

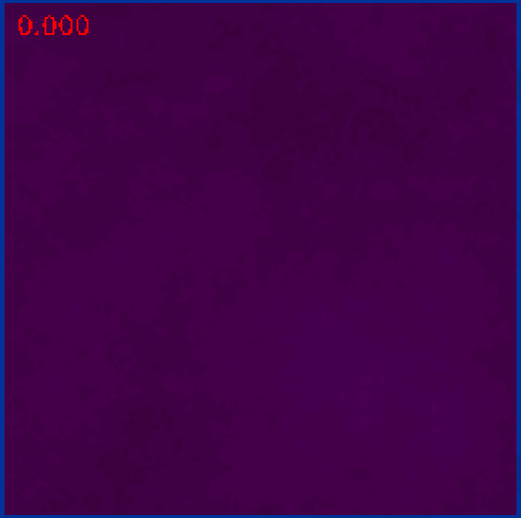
For Q >> 1: coefficients of Q^2 must cancel → Lindblad resonances (or k → imaginary everywhere!)

$$\kappa^2 - Q^2\kappa^2 + m^2Q^2(\Omega - \Omega_p)^2 \quad \Rightarrow \quad -\kappa^2 + m^2(\Omega - \Omega_p)^2 = 0$$

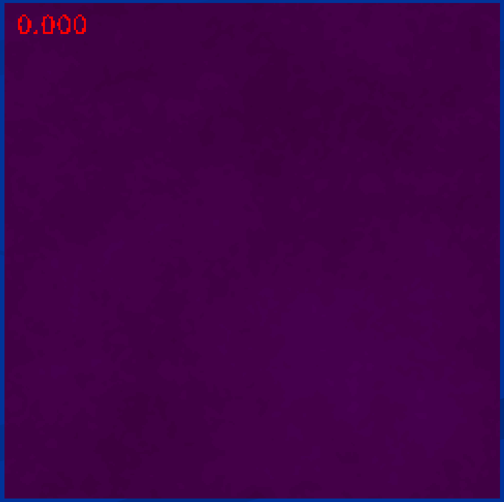
Examples:



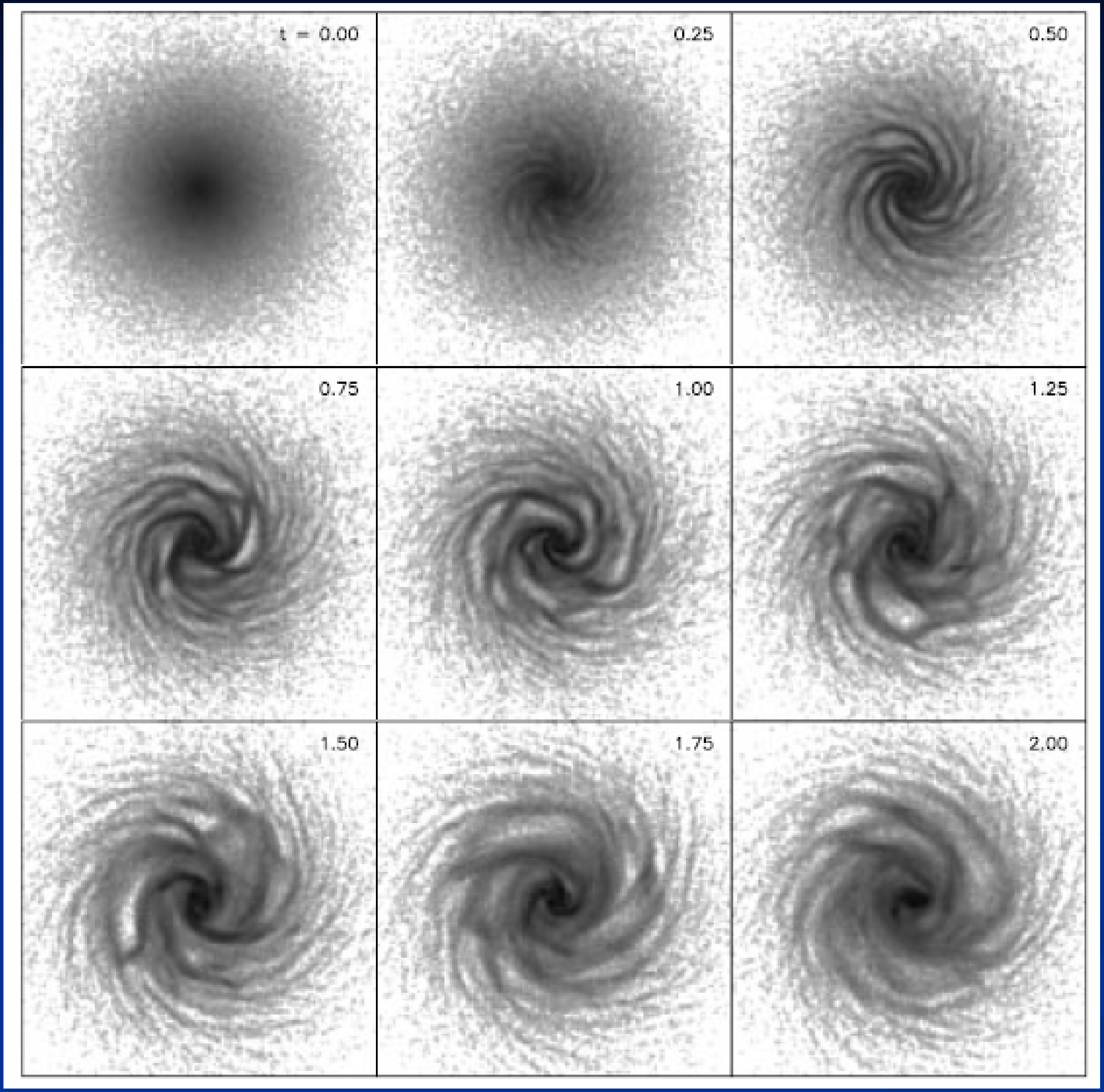
Q=0.8



Q=1.1



Q=1.3



Q = 0.63

- Wave group velocity

dispersion relation for sound waves (G = 0): $\omega^2 = c_s^2 k^2$

$\rightarrow$ $\omega(k) = \pm c_s k$

dispersion relation for sound waves (with gravity):

$$\omega^2(k) = c_s^2 k^2 - 4\pi G\rho = c_s^2 (k^2 - 4\pi G\rho / c_s^2)$$

$\rightarrow$ dispersive medium

Option 1: uniform wave train ← linear combination of plane waves

→ unphysical, because infinite waves

Option 2: spatially localized wave: $f(x, t) = \int_{-\infty}^{\infty} F(k)\, e^{i[kx - w(k)t]}\, dk$

(wave packet around k_0)

Taylor expansion around k_0: $\omega(k) = \omega(k_0) + (k - k_0) v_g$

where $v_g = \left[\frac{d\omega(k)}{dk} \right]_{k_0}$

•Wave group velocity

Wave packet propagation in a dispersive medium (when ω[k]):

$$v_g = \frac{d\omega(k)}{dk}$$

In a disk:

$$v_g = \frac{\partial\omega(k,r)}{\partial k}$$

Take the dispersion relation for a gaseous disk:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma |k| + k^2 c_s^2$$

Take a derivative with respect to “k”:

$$v_g(r) = \mathrm{sign}(k)\frac{\pi G\Sigma - |k| c_s^2}{m(\Omega - \Omega_p)} = \mathrm{sign}(k)\frac{(c_s\kappa/Q) - |k| c_s^2}{m(\Omega - \Omega_p)}$$

Wave phase velocity: $v_g = \frac{\omega}{k}$ → equal to **group** velocity only in non-dispersive media