GALACTIC DYNAMICS and ISM *Spring 2014*
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PROBLEM SET 2

Problem 1 (Disk potential)
The potential of a Kuzmin disk is given in cylindrical polar coordinates r, z by

$$\Phi(r,z) = -\frac{GM}{\sqrt{r^2 + (a+|z|)^2}}.$$

(a). Does this potential correspond to that of a point? If *yes*, give the location of this point.
(b). Show that $\Delta\Phi = 0$ everywhere except at $z = 0$.
(c). Use the divergence theorem to find the surface density of Kuzmin disk.

Problem 2 (Virial theorem)
(a). If an isolated cluster of stars is initially in equilibrium, and a fraction f of its stars is suddenly removed at each radius, by what factor do the kinetic (E_k) and potential (E_p) energies change?
(b). If the initial potential energy is E_{p0}, what is the starting energy of the cluster E_0? What is the resulting energy of the cluster, E_1?
(c). If $f > 0.5$, the stars are no longer in equilibrium. Show that when the remaining stars come to a new equilibrium, the average distance between them changes and by what factor.

Problem 3 (Gauss Law)

(a). Use Gauss' law to derive an expression for the gravitational force in the z direction due to an infinite sheet of surface density Σ lying in the $x-y$ plane.

(b). A star has velocity 30 km s^{-1} perpendicular to the galactic plane as it crosses the plane, and is observed to have a maximal departure above the plane of 500 pc. Approximating the disk as an infinite gravitating sheet of matter, estimate its surface density Σ in $M_\odot$ pc^{-2}.

Problem 4 (Gravothermal Catastrophe)

Consider a system in which the particles interact through the potential $\Phi = A|\mathbf{r}|^{-\alpha}$, where A is a constant.

(a). Show, that for such a system, the virial theorem has the form

$$2K + \alpha W = 0,$$

where K is the kinetic energy and W is the potential energy.

(b). For what values of α does the system have a negative heat capacity?

Problem 5 (Jeans Equations)

(a). Use the divergence theorem to find the gravitational potential at the height z above a 2-D uniform sheet of matter with surface density Σ.

(b). Show that the vertical force does not depend on z, and find $\Delta\Phi$.

(c). Suppose (as above) the mass of the Galaxy was all in a flat uniform disk. Write the equation of the vertical (z) equilibrium and use it to find the density $n(z)$ of stars, assuming they have a constant velocity dispersion σ_z. As in the Earth atmosphere, where the acceleration of gravity is also nearly independent of height, show that $n(z)$ drops by a factor of e as $|z|$ increases by $h_z = \sigma_z^2/2\pi G\Sigma$. Estimate h_z near the Sun, taking $\sigma_z = 20$ km s^{-1}.

Problem 6 (Jeans Equations)

A spherical elliptical galaxy has a total density distribution

$$\rho_{\rm tot}(R) = \frac{\rho_0}{1 + R^2/a^2},$$

as a function of radial distance R from its center, where ρ_0 and a are constants (here the total density means the density including all stars, gas and dark matter).

(a). Show that the total mass $M(R)$ interior to a radius R has the form $M(R) \propto R^3$ for $R << a$ and $M(R) \propto R$ for $R >> a$.

(b). Consider a population of massless test particles in the potential of this galaxy. Assume that this population is spherical, non-rotating, isothermal and isotropic, with velocity dispersion σ in each velocity component. What is the radial density distribution $\rho_{\rm p}(R)$ of this test particle population, expressed in terms of $M(R)$ and R?

(c). Solve for $\rho_{\rm p}(R)$ in terms of R explicitly for large radii (i.e. for regions where $R >> a$) to show that the density has a power law dependence on radius. What is the index of this power law?

(d). Give a physical interpretation of this index. What is the condition for the density distributions of the test particle population and the galaxy itself to have similar forms at large R?

Problem 7 (Stellar Encounters)

The deflecting component of stellar velocity $\Delta v_\perp$ is steadily increasing in multiple weak encounters. After time t, its square is given by

$$< \Delta v_\perp^2 > = \int_{b_{min}}^{b_{max}} nvt \left(\frac{2Gm}{bv}\right)^2 2\pi b db,$$

where all the variables and parameters have been defined in class.

(a). Computer simulations of galactic disks often confine all the particles to a single plane: instead of a volume density of stars we have a surface (number) density Σ. The term $2\pi b db$ in the above equation (for a 3-D case) is replaced by $2db$. Why?

(b). Show that now t_{relax} does not depend on Λ, but only on b_{min}, and that $t_{relax}/t_{cross} = v^2/8GRm\Sigma$, where R is the radius of the stellar system.

(c). If the mass density $m\Sigma$ is fixed, this ratio is independent of the number of simulation particles. How can this happen?