GALACTIC DYNAMICS and ISM *Spring 2014*
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PROBLEM SET 3

Problem 1 (Stellar Motion)

A particle moves in circular orbit in a potential $\Phi(r) = -Kr^{-\alpha}$, where K and α are positive constants.

(a). Show that particle's orbital velocity is given by $v^2(r) = -\alpha\Phi(r)$.

(b). Two gas clouds, masses m_1, m_2, follow circular orbits at radii r_1, r_2, with $r_1 < r_2$. What is the total energy E, and angular momentum L?

(c). The gas clouds are now displaced to different circular orbits at radii $r_1 + \Delta r_1, r_2 + \Delta r_2$. How must $\Delta r_1, \Delta r_2$ be related so that L is unchanged? Assuming $\Delta r_1, \Delta r_2$ are small, what is the energy change ΔE?

(d). Show that if $\alpha < 2$, the angular momentum $rv(r)$ of a circular orbit increases with r. (This is condition for the circular orbit to be stable!) In that case, show that the energy of the second state is lower than the initial energy if $\Delta r_1 < 0$.

Problem 2 (Jeans Instability)

The Jeans instability can be analyzed exactly, without invoking the Jeans swindle, in certain cylindrical rotating systems. Consider a homogeneous, self-gravitating fluid of density ρ_0, contained in an infinite cylinder of radius R_0. The cylinder walls and the fluid rotate at angular speed $\mathbf{\Omega} = \Omega\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ lies along the axis of the cylinder.

(a). Show that the gravitational force per unit mass inside the cylinder is

$$-\bigtriangledown\Phi_0 = -2\pi G\rho_0(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y).$$

(b). Using Euler's equation for a uniformly rotating gas sheet (see class notes), find the condition on Ω so that the fluid is in equilibrium with no pressure gradients.

(c). Now let $R_0 \to \infty$, or, what is equivalent, consider wavelengths $\lambda << R_0$, so that the boundary condition due to the wall can be neglected. Working in the rotating frame, find the dispersion relation analogous to the one we found for Jeans instability ($\omega^2 = c_s^2k^2 - 4\pi G\rho_0$) for

(i) waves propagating perpendicular to $\mathbf{\Omega}$; and

(ii) waves propagating parallel to $\mathbf{\Omega}$.

Show that the waves propagating perpendicular to $\mathbf{\Omega}$ are always stable, while waves propagating parallel to $\mathbf{\Omega}$ are stable if and only if the usual Jeans criterion ($k^2 < k_J^2 \equiv ...$) is satisfied.

Problem 3 (Lindblad Resonances)

Suppose that the angular velocity in the Milky Way galaxy is given by

$$\Omega(r) = \frac{v_{\rm c}}{r},$$

where $v_{\rm c} = 250$ km s^{-1}. Taking a two-armed perturbation, find the locations of the Lindblad resonances. How do these values compare with that known about the spiral structure of the MW?

Problem 4 (Density Waves in Gaseous Disks)

Show that the group velocity of density waves in a gaseous disk with $Q = 1$ is equal (within a sign) to the sound speed.

Problem 5 (Bending Waves)

So-called bending waves in disk galaxies have the dispersion relation

$$(m\Omega - \omega)^2 = \nu^2 + 2\pi G\Sigma|k|,$$

where ν is epicyclic frequency in the z-direction.

(a). Find the group velocity of bending waves.

(b). For retrograde ($\Omega_p < 0$) bending waves with $m = 1$ find which waves propagate inward (leading or trailing), and which propagate outward (leading or trailing).

Note: you can read more about the bending waves in BT Section 6.6 (pp. 540-548).