Ex 4-1.

(a) $|E_1\rangle$ (Ground state).

\[ x = \frac{1}{2} \lambda = a \Rightarrow \lambda = 2a. \]
\[ k = \frac{2\pi}{2a} = \frac{\pi}{a}. \]
\[ : E_1 = \frac{\hbar^2}{2m} \left( \frac{\pi}{a} \right)^2 = \frac{\hbar^2}{2m} \left( \frac{\lambda}{a} \right)^2. \]

\[ |E_1\rangle = \sqrt{\frac{2}{a}} \cos \frac{\pi x}{a}. \]

(b) $|E_2\rangle$ (1st excited state).

\[ \lambda = a. \Rightarrow k = \frac{2\pi}{a}. \]
\[ : E_2 = \frac{\hbar^2}{2m} \left( \frac{2\pi}{a} \right)^2 = \frac{2\hbar^2}{m} \left( \frac{a}{a} \right)^2. \]

\[ |E_2\rangle = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}. \]

(c) $|E_3\rangle$ (2nd excited state).

\[ \frac{3}{2} \lambda = a. \Rightarrow \lambda = \frac{2}{3} a. \]
\[ k = \frac{2\pi}{\frac{2}{3} a} = \frac{3\pi}{2a} = \frac{8\pi}{a}. \]
\[ : E_3 = \frac{\hbar^2}{2m} \left( \frac{3\pi}{2a} \right)^2 = \frac{9\hbar^2}{2m} \left( \frac{a}{a} \right)^2. \]

\[ |E_3\rangle = \sqrt{\frac{2}{a}} \cos \frac{3\pi x}{a}. \]
(c) $|\Psi_2\rangle$:

\[ P_2 = |\Psi_2(x)|^2. \]

\[
\begin{align*}
\langle X_2 \rangle &= \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} x \sin \frac{2\pi x}{a} \, dx = 0. \\
\langle x_3 \rangle &= \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} x \cos \frac{2\pi x}{a} \, dx = 0. \\
\langle h \rangle &= \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin \frac{2\pi x}{a} \left( -ih \frac{d}{dx} \right) \cos \frac{2\pi x}{a} \, dx = 0. \\
\langle P_2 \rangle &= \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \frac{2\pi x}{a} \left( -ih \frac{d}{dx} \right) \cos \frac{2\pi x}{a} \, dx = 0. \\
\end{align*}
\]

(result obvious!)
(c) \( |E_2>: \)

\[
\langle x^2 \rangle = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 \sin \frac{2\pi x}{a} \, dx.
\]

From integration table,

\[
\langle x^2 \rangle = \frac{2}{a} \left[ \frac{x^2}{6} - \left( \frac{x^2}{4} \right) - \frac{1}{8} \frac{2\pi}{a} \right] \sin \frac{2\pi x}{a} - \frac{x \cos \frac{2\pi x}{a}}{4 \left( \frac{2\pi}{a} \right)^2} \right]_{-\frac{a}{2}}^{\frac{a}{2}}
\]

\[
= \frac{2}{a} \left[ \frac{a^2}{6} - 2 \right]
\]

\[
= \frac{2}{a} \left[ \frac{a}{6} \right] - \frac{a}{2} \frac{2}{4 \left( \frac{2\pi}{a} \right)^2}
\]

\[
= \frac{2}{a} \left[ \frac{a^2}{24} - \frac{a^3}{16\pi^2} \right]
\]

\[
= a^2 \left( \frac{1}{12} - \frac{1}{8\pi^2} \right).
\]

1. \( \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \)

\[
= \sqrt{\langle x^2 \rangle}
\]

\[
= a \sqrt{\frac{1}{12} - \frac{1}{8\pi^2}}
\]

\[
\langle p^2 \rangle = \langle mE \rangle = 2m \langle E \rangle = 2m \frac{\hbar^2 \left( \frac{a}{12} \right)^2}{m (\frac{a}{12})}
\]

\[
= 4\hbar^2 \frac{a^2}{12}.
\]

2. \( \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}. \)

\[
= \sqrt{\langle p^2 \rangle}
\]

\[
= 2\hbar \frac{\pi}{\alpha}
\]

3. \( \Delta x \Delta p = a \sqrt{\frac{1}{12} - \frac{1}{8\pi^2}} \cdot 2\hbar \frac{\pi}{\alpha} = \frac{2\pi}{\alpha} \sqrt{\frac{1}{12} - \frac{1}{8\pi^2}} + \frac{\hbar}{a} \quad \approx 0.356 \hbar = 5.83 \hbar. \)
\[ |E_3\rangle: \]
\[ \langle x^2 \rangle = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 \cos \frac{6\pi x}{a} \, dx. \]

From integration table:
\[ \langle x^2 \rangle = \frac{2}{a} \left[ \frac{x^3}{6} + \left( \frac{2}{3} \right) \frac{1}{8} \sin \left( \frac{6\pi x}{a} \right) + \frac{x \cos \frac{6\pi x}{a}}{4 \left( \frac{8\pi}{a} \right)^2} \right]_{-\frac{a}{2}}^{\frac{a}{2}}. \]

\[ = \frac{2}{a} \left[ \frac{1}{6} \left( \frac{a}{2} \right)^3 + \frac{1}{8} \frac{1}{16} \left( \frac{8\pi}{a} \right)^2 \right] \]

\[ = \frac{2}{a} \left[ \frac{a^2}{24} - \frac{a^2}{36\pi^2} \right] \]

\[ = a^2 \left( \frac{1}{12} - \frac{1}{18\pi^2} \right) \]

\[ \langle p^2 \rangle = 2m \langle E \rangle = 2m \cdot \frac{9\hbar^2}{2m} \left( \frac{\pi}{a} \right)^2 = 9 \left( \frac{\pi}{a} \right)^2 \hbar^2. \]

\[ \Delta p = 3 \frac{\pi}{a} \hbar \]

\[ \Delta x \Delta p = a \sqrt{\frac{1}{12} \frac{1}{18\pi^2} \frac{3\pi}{a} \hbar} \]

\[ = 3\pi \sqrt{\frac{1}{12} \frac{1}{18\pi^2} \hbar} \]

\[ \approx 2.63 \hbar. \]