INSTRUCTIONS
1) Wait for oral instructions before starting the test.
2) Remember to justify (in English) as many steps as possible for partial credit.
3) No calculators or other aids permitted.

For the graders:

1. 
2. 
3. 
4. 

TOTAL
\[
\sin \theta = o/h \\
\cos \theta = a/h \\
\tan \theta = o/a
\]

\[
\sin(90° - \theta) = \cos \theta \quad \cos(90° - \theta) = \sin \theta
\]

\[
\begin{align*}
\sin 0 &= 0 & \cos 0 &= 1 \\
\sin 90° &= 1 & \cos 90° &= 0 \\
\sin 30° &= 1/2 & \cos 60° &= 1/2
\end{align*}
\]

\[
2 \sin \theta \cos \theta = \sin 2\theta
\]
1. Circular motion I

a) i) A circle has radius $r$. If the arc length of a certain segment of the circle is $2r$, how many radians is the segment angle? [2 points]

\[ 2 \]

ii) How many radians is one complete revolution? [2 points]

\[ 2 \pi \]

b) Write down the rotational equivalent of the linear-motion displacement equation $v = v_0 + at$. [4 points]

\[ \omega = \omega_0 + \alpha t \]

\[ \text{(4)} \]

c) Write down the rotational equivalent of the linear-motion displacement equation $x = v_0 t + \frac{1}{2} at^2$. [4 points]

\[ \Theta = \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \text{(4)} \]
1. d) A disc, starting from rest, undergoes a constant angular acceleration for a time \( t_1 \) and reaches an angular velocity of \( \omega_1 \). It then undergoes a constant negative acceleration and comes to rest in a time \( t_2 \). Find an expression, in terms of the quantities \( t_1 \), \( t_2 \), and \( \omega_1 \) for the total number of revolutions of the disc in the time \( t_1 + t_2 \). [13 points]

We need to find total angular displacement \( \theta \) and divide by \( 2\pi \) to find number of revolution.

First part:

\[
\begin{align*}
\omega_1 &= 0 + \omega_1 t_1 \quad \therefore \quad \dot{\omega}_1 = \omega_1 / t_1 \\
\dot{\theta}_1 &= \frac{1}{2} \omega_1 t_1^2 = \frac{1}{2} \omega_1 t_1
\end{align*}
\]

[Or \( \theta_1 = \omega_{\text{average}} t_1 = \frac{0 + \omega_1 t_1}{2} = \frac{1}{2} \omega_1 t_1 \)]

Second part:

\[
\begin{align*}
\Theta &= \omega_1 + \omega_2 t_2 \quad \therefore \quad \dot{\omega}_2 = -\frac{\omega_1}{t_2}
\end{align*}
\]

\[
\begin{align*}
\dot{\theta}_2 &= \omega_1 t_2 + \frac{1}{2} \omega_2 t_2^2 = \omega_1 t_2 - \frac{1}{2} \frac{\omega_1}{t_2} t_2^2 = \frac{1}{2} \omega_1 t_2
\end{align*}
\]

[Or \( \theta_2 = \omega_{\text{average}} t_2 = \frac{\omega_1 + 0}{2} t_2 = \frac{1}{2} \omega_1 t_2 \)]

\[
\begin{align*}
\theta &= \theta_1 + \theta_2 = \frac{1}{2} \omega_1 (t_1 + t_2)
\end{align*}
\]

\[
\text{Number of revolution} = \frac{\omega_1}{2\pi} (t_1 + t_2)
\]
2. Circular motion II and Gravity

a) An object of mass \( m \) is travelling at constant speed \( v \) in a circle of radius \( r \). Write down an expression for the magnitude of the force required for this circular motion. [4 points]

\[
F = \frac{m v^2}{r}
\]

b) Write down the force \( F \) due to the gravitational attraction between two point masses \( m_1 \) and \( m_2 \) separated by a distance \( r \) (use the symbol \( G \) for the universal constant). [4 points]

\[
F = G \frac{m_1 m_2}{r^2}
\]

c) The answer to (b) is still valid if the point masses are replaced by spheres. From where to where is \( r \) measured in this case? [4 points]

Between their centers
2. d) A non-rotating spherical planet has radius $R$ and uniform density $\rho$ (mass per unit volume). A small rock is orbiting the planet just above the surface (i.e., at $R$; there is no atmosphere to slow it down!). Find an expression for the orbital period $T$ of the rock. Express your answer in terms of $\rho$, and $G$. (Your answer will not contain $R$). [13 points] [HINTS: Volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$; Mass = volume $\times$ density]

Let mass of rock = $m$, let mass of planet = $M$

Gravitational attraction on rock = $F = G \frac{mM}{R^2}$

This provides centripetal force $\frac{mv^2}{R}$

$G \frac{mM}{R^2} = \frac{mv^2}{R}$ or $GM = v^2 R$ \hspace{1cm} (1)

1 Orbit = $2\pi R$. Time for one orbit is $T = \frac{2\pi R}{v}$

Thus $T^2 = \frac{(2\pi)^2 R^2}{v^2} = \frac{(2\pi)^2 R^3}{GM}$ from (1).

Using "HINTS": $M = \frac{4}{3}\pi R^3 \times \rho$

Thus $T^2 = \frac{(2\pi)^2 R^3}{\frac{4}{3}\pi R^3 \rho G} = \frac{3\pi}{G \rho}$

$\therefore T = \sqrt{\frac{3\pi}{G \rho}}$ \hspace{1cm} (13)
3. Equilibrium.

a) Write down the torque \( \tau \) about an axis \( A \), if a force \( F \) acts at a point \( B \) a distance \( d \) from the axis. The force acts in a direction perpendicular to the line \( AB \). [4 points]

\[ \tau = Fd \]

b) Write down the torque \( \tau \) about an axis \( A \), if a force \( F \) acts at a point \( B \) a distance \( L \) from the axis. The force acts in a direction \( \theta \) with respect to the line \( AB \). [4 points]

\[ \tau = FL \sin \theta \]

c) Write down (in mathematical notation) the two conditions for equilibrium. [2 points each]

\[ \sum F = 0 \]
\[ \sum \tau = 0 \]
3. d) A non-uniform rod, of mass $M$ and length $\ell$, has its center of gravity located a distance $a$ from the left end. Find expressions for the forces $F_L$ and $F_R$ that need to be applied at the left and right hand ends of the rod in order to keep the rod horizontal and above the ground. [13 points]

\[ \sum F = 0 \text{ gives } F_L + F_R = Mg \]

Could take $\sum \tau = 0$ about (1) Left end, (2) Center, or (3) Right end. Which is best?

In fact, Left end is easiest:

\[ \sum \tau = 0 : \quad Mg \alpha = F_R \ell \]

Then \[ F_R = \frac{Mg \alpha}{\ell} \]

\[ \text{gives } \quad F_L = Mg - F_R = \frac{Mg (1 - \frac{a}{\ell})}{\ell} \]

\[ \boxed{13} \]
4. Rotational Dynamics.

a) Write down the form of Newton’s second law of motion applicable to rotational motion (in the form \( \tau = \cdots \)), and define each symbol in words. [4 points]

\[ \tau = I \alpha \]

\( \tau = \text{Torque} \quad I = \text{Moment of Inertia} \quad \alpha = \text{angular acceleration} \)

b) A massless rod of length \( L = 6 \text{m} \) has a mass \( m = 10 \text{kg} \) fixed at one end and an equal mass \( m = 10 \text{kg} \) fixed at the other end. What is the moment of inertia about an axis perpendicular to the rod and through its center? [4 points]

\[ I = \sum m r^2 = 2 \times 10 \times 3^2 = 20 \times 9 = 180 \text{ kg m}^2 \]

c) A massless rod of length \( L = 6 \text{m} \) has a mass \( m = 10 \text{kg} \) fixed at one end and an equal mass \( m = 10 \text{kg} \) fixed at the other end. What is the moment of inertia about an axis perpendicular to the rod and through one of its ends? [4 points]

\[ I = 10 \times 6^2 = 360 \text{ kg m}^2 \]
4. d) **Rotational Dynamics.**

A cylinder with moment of inertia \( I_1 = 1 \text{ kg.m}^2 \) rotates with angular velocity \( \omega_1 = 6 \text{ rad/s} \) about a frictionless vertical axle. See “initial” diagram below. A second cylinder, with moment of inertia \( I_2 = 2 \text{ kg.m}^2 \), initially not rotating, drops onto the first cylinder. Since the surfaces are rough, the two eventually reach the same angular velocity \( \omega_2 \), as shown in “intermediate” below. Next, a third cylinder, with moment of inertia \( I_3 = 3 \text{ kg.m}^2 \), initially not rotating, drops onto the first two to produce the “final” result shown below, with all three rotating at the same angular velocity \( \omega_3 \).

(i) What is the moment of inertia of the “final” system? [3 points]

\[
I = I_1 + I_2 + I_3 = 1 + 2 + 3 = 6 \text{ kg.m}^2
\]

(ii) What are the “intermediate” and “final” angular velocities, \( \omega_2 \) and \( \omega_3 \)? [5 points]

\[
I_1 \omega_1 = (I_1 + I_2) \omega_2 \quad \text{so} \quad \omega_2 = \frac{I_1}{I_1 + I_2} \omega_1 = \frac{1}{1 + 2} \cdot 6 = 2 \text{ rad/s}
\]

\[
\omega_3 = \frac{I_1}{I_1 + I_2 + I_3} \omega_1 = \frac{1}{3} \text{ rad/s}
\]

(iii) What is the ratio of the total initial to the total final kinetic energy? [5 points]

\[
KE_i = \frac{1}{2} I \omega_1^2 = \frac{1}{2} \times 1 \times 6^2 = 18 \text{ J}
\]

\[
KE_f = \frac{1}{2} (I_1 + I_2 + I_3) \omega_3^2 = \frac{1}{2} \times 6 \times 1^2 = 3 \text{ J}
\]

\[
\frac{KE_i}{KE_f} = 6 \quad \text{(or} \quad \frac{KE_f}{KE_i} = \frac{1}{6})
\]