Method to calculate the covariant uncertainty in the physics asymmetry for the n3He experiment

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1 Simulation Summary

The current calculation is a complete monte carlo simulation of the detector signals from the neutron beam to the output wires. Previous versions of this simulation were presented in March and November 2014 (available on the n3He wiki). Briefly, the simulation involves these steps:

-Beam monitor data is fitted and used to generate the neutron velocity distribution.
-Neutron beam scans are fitted and used to generate the coordinates of the neutron events.
-ENDF cross section data is fitted and used to determine reaction depths.
-Reaction product energy depositions are derived from PSTAR data.
-Reaction product times and cells are tracked for 7056 signal elements.
-Statistics are calculated using the following method:

2 Element Asymmetry

Every neutron event i generates a unique angle θ_i, which is the angle between the proton and neutron momenta. The simulation calculates quantities for K elements, indexed by κ. For each neutron event, the energies deposited into each element, Q_i^κ, are recorded. The total energy received by an element is the sum of all the energies deposited in each event:

$$E^\kappa = \sum_{i=1}^{N_{mc}} Q_i^\kappa \tag{1}$$

Let us define the *expectation value* for the energy in cell κ as $\langle E^\kappa \rangle$:

$$\langle E^\kappa \rangle = \langle \sum_{i=1}^{N_{mc}} Q_i^\kappa \rangle$$

We will analyze how to evaluate this summation further in section 3. For now, let us define our observable quantities. This expectation value corresponds to our observable, since for a large number of trials, the average simulated value represents the observed energy in an element. For a given helicity h, the experimentally observed yield for cell κ, Y_h^κ, is

$$Y_h^\kappa = \langle E^\kappa (1 + h\alpha \cos\theta) \rangle, \tag{2}$$

where α is the size of the underlying physics asymmetry. From this, we can find a relation between our observable quantities and the simulated ones:

$$\frac{Y_+^\kappa - Y_-^\kappa}{Y_+^\kappa + Y_-^\kappa} = \alpha_\kappa \frac{\langle E^\kappa \cos\theta \rangle}{\langle E^\kappa \rangle}$$

Define the geometry factor for element κ as:

$$G_\kappa = \frac{\langle E^\kappa \cos\theta \rangle}{\langle E^\kappa \rangle} = \frac{\sum_{i=1}^{N_{mc}} Q_i^\kappa \cos\theta_i}{\sum_{i=1}^{N_{mc}} Q_i^\kappa} \tag{3}$$

Then the element κ produces an experimental value of the physics asymmetry:

$$\alpha_\kappa = \frac{1}{G_\kappa}\frac{Y_+^\kappa - Y_-^\kappa}{Y_+^\kappa + Y_-^\kappa} \tag{4}$$

3 Statistics

There are 144 physical cells, and 49 time slices, for a total of 7056 signal elements (although we will treat events occuring at different times as relatively independent). The probability that a single neutron event will deposit energy into an element is much smaller than one, even for elements at the front of the chamber. Because of this, the binomial distribution of the number of deposition events n_κ can be approximated with Poisson statistics. Since the probability of any individual element receiving energy from a given neutron event is much less than one, and each pulse will contain 10^8 neutrons, we can assume that our summation $\sum_n^{N_{mc}} \simeq \sum_n^{\infty}$. We can introduce the counting statistics by taking advantage of this property.

Now we return to evaluating (1). The expectation value of the observed energy E^κ is defined as the average value obtained from repeated trials. Let us take advantage of the Poisson nature of the underlying quantities to simplify this procedure. The expectation value of the element signal can be expressed in terms of the average number of depositions in an element and the expectation value of the deposition energy in element κ.

$$E^\kappa = \sum_n^{N_{mc}} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!}\sum_{i=1}^{n} Q_i^\kappa \tag{5}$$

The index n is a Poisson-distributed random variable of the number of depositions occuring in element κ, and Q_i^κ is a random variable with a corresponding distribution over the range of possible energies deposited in element κ. Now, apply the expectation value to this summation:

$$\langle E^\kappa\rangle = \langle\sum_n^{N_{mc}} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!}\sum_{i=1}^{n} Q_i^\kappa\rangle = \sum_n^{N_{mc}} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!}\langle\sum_{i=1}^{n} Q_i^\kappa\rangle$$

The expectation operator applies to every term in the summation, so we can move it to just the sum over i. Now, there will be a large number of neutron events in a simulation, so we can assume that the average of the distribution of all possible deposition energies in element κ will be the same as the average of the energies in a single simulation. This will be used to approximate one pulse, so call it the *pulse average*. For clarity, denote the total pulse average with a bar:

$$\langle\sum_{i=1}^{n} Q_i^\kappa\rangle = n\overline{Q^\kappa}$$

By Poisson statistics, the quantity λ_κ is equal to the average number of deposition events occuring in element κ. Use the same notation and call this the pulse average of $\bar{n}_\kappa$, where we will assume again that the number of events in a simulation is large enough that this local average converges to the expectation value. The sum over n gives the result

$$\langle E^\kappa\rangle = \bar{n}_\kappa\overline{Q^\kappa} \tag{6}$$

Here we have used the fact that

$$\sum_{n}^{\infty} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!} n = \bar{n}_\kappa$$

We have now expressed the expectation value of the observable energy in terms of two averages which we can measure from a single simulation. This result is intuitively obvious, but we will apply the same method to calculate some others which are not in section 4.

To calculate the statistics of the running time, as well as optimizing experimental parameters, we can calculate the uncertainty in the physics asymmetry. To do this, consider the covariance of the asymmetries of two elements (as defined in (4)):

$$\sigma(\alpha_\kappa, \alpha_\beta) = \sigma_{\alpha_\kappa \alpha_\beta} = \sum_{h_1}^{\pm 1} \sum_{h_2}^{\pm 1} \frac{\partial \alpha_\kappa}{\partial Y_{h_1}^\kappa} \frac{\partial \alpha_\beta}{\partial Y_{h_2}^\beta} \sigma_{Y_{h_1}^\kappa Y_{h_2}^\beta} \tag{7}$$

The indices h_1 and h_2 refer to the difference helicity states of the neutron beam. Events that occur in different helicity states will not be correlated, so the signal covariances are zero unless $h_1 = h_2$. We can rewrite the covariance in the uncertainty as:

$$\sigma_{\alpha_\kappa \alpha_\beta} = \frac{\partial \sigma_\kappa}{\partial Y_+^\kappa} \frac{\partial \sigma_\beta}{\partial Y_+^\beta} \sigma_{Y_+^\kappa Y_+^\beta} + \frac{\partial \sigma_\kappa}{\partial Y_-^\kappa} \frac{\partial \sigma_\beta}{\partial Y_-^\beta} \sigma_{Y_-^\kappa Y_-^\beta}$$

In addition, since the signal yields for different helicity states are almost identical, the energy variances for the different helicity states are equal: $\sigma_{Y_+^\kappa Y_+^\beta} = \sigma_{Y_-^\kappa Y_-^\beta} = \sigma_{Y^\kappa Y^\beta}$. We can further simplify the expression for the asymmetry covariance in the following way:

$$\sigma_{\alpha_\kappa \alpha_\beta} = \left[\frac{\partial \sigma_\kappa}{\partial Y_+^\kappa} \frac{\partial \sigma_\beta}{\partial Y_+^\beta} + \frac{\partial \sigma_\kappa}{\partial Y_-^\kappa} \frac{\partial \sigma_\beta}{\partial Y_-^\beta} \right] \sigma_{Y^\kappa Y^\beta}$$

Carrying out the partial derivatives and simplifying, we get:

$$\sigma_{\alpha_\kappa \alpha_\beta} = \frac{1}{G_\kappa G_\beta} \left[\frac{4(Y_+^\kappa Y_+^\beta + Y_-^\kappa Y_-^\beta)}{(Y_+^\kappa + Y_-^\kappa)^2 (Y_+^\beta + Y_-^\beta)^2} \right] \sigma_{Y^\kappa Y^\beta}$$

Due to the high statistics in each pulse, we have $Y_+^\kappa = Y_-^\kappa = \langle E^\kappa \rangle = \bar{n}_\kappa \overline{Q^\kappa}$:

$$\sigma_{\alpha_\kappa \alpha_\beta} = \frac{1}{2 G_\kappa G_\beta} \frac{\sigma_{Y^\kappa Y^\beta}}{\bar{n}_\kappa \overline{Q^\kappa} \bar{n}_\beta \overline{Q^\beta}} \tag{8}$$

Note that the two in the denominator comes from the fact that two pulses are necessary to calculate α_κ. To evaluate this, we need to define the quantity $\sigma_{Y^\kappa Y^\beta}$, which is the covariance of two signal energies:

$$\sigma_{Y^\kappa Y^\beta} = \langle Y^\kappa Y^\beta \rangle - \langle Y^\kappa \rangle \langle Y^\beta \rangle = \langle E^\kappa E^\beta \rangle - \langle E^\kappa \rangle \langle E^\beta \rangle + \alpha h[...] + \alpha^2[...]$$

Each of the observable energies has a correction term on the order of α or α^2 (2). For this experiment, our asymmetry is on the order of 10^{-7}. Any terms in the above expression with one or more coefficients of α will be 10^7 times smaller than the rest, so we can safely neglect them:

$$\sigma_{Y^\kappa Y^\beta} = \langle E^\kappa E^\beta \rangle - \langle E^\kappa \rangle \langle E^\beta \rangle \tag{9}$$

The solution in (6) expresses the product on the right in terms of pulse averages. Now we consider the term on the left:

$$\langle E^\kappa E^\beta \rangle = \Big\langle \sum_n^{N_{mc}} \frac{e^{-\lambda}\lambda^n}{n!} \sum_i^n Q_i^\kappa \sum_j^n Q_j^\beta \Big\rangle \tag{10}$$

In order to evaluate this, we must consider the different event subsets in the Poisson sum. Specifically, we must treat the difference between events which deposit energy coincidentally in both elements κ and β (leading to energy correlations) and those which deposit energy only in one. Each of these subsets will have its own separate Poisson parameter. For example, we can rewrite (6) using this decomposition:

$$\begin{aligned}\langle E^\kappa \rangle &= \sum_n^{N_{mc}} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!} \Big\langle \sum_i^n Q_i^\kappa \Big\rangle = \sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!} \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\kappa}}}\lambda_{\acute{\kappa}}^n}{n!} \Big\langle \sum_i^{n_{\kappa\beta}} Q_i^\kappa + \sum_j^{n_{\acute{\kappa}}} Q_j^\kappa \Big\rangle \\ &= \sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!} \langle \sum_i^{n_{\kappa\beta}} Q_i^\kappa \rangle + \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\kappa}}}\lambda_{\acute{\kappa}}^n}{n!} \Big\langle \sum_j^{n_{\acute{\kappa}}} Q_j^\kappa \Big\rangle = \acute{n}_\kappa \acute{Q^\kappa} + \widehat{n}_\kappa \widehat{Q^\kappa} = \bar{n}_\kappa \overline{Q^\kappa}\end{aligned} \tag{11}$$

The acute denotes events which are unique to the element, and the caret denotes events which are coincident with another cell (unspecified here). This shows that adding the two subsets of depositions gives the total sum of depositions for all events, as expected. Evaluating sums of this form implies the separation of the sum over events into distinct subsets:

$$\sum_i^n Q_i^\kappa = \sum_i^{n_{\kappa\beta}} Q_i^\kappa + \sum_j^{n_{\acute{\kappa}}} Q_j^\kappa + \sum_k^{n_{\acute{\beta}}} Q_k^\kappa = \sum_i^{n_{\kappa\beta}} Q_i^\kappa + \sum_j^{n_{\acute{\kappa}}} Q_j^\kappa \tag{12}$$

The third term will be zero, since depositions in element κ will not appear in any sum over events unique to element β. So we see that for every term in the Poisson sum, the number of depositions, n, will be distributed between coincident and unassociated events. Now we can continue evaluating (10), using the decompositions from (11) and (12):

$$\langle E^\kappa E^\beta \rangle = \Big\langle \sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!} \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\kappa}}}\lambda_{\acute{\kappa}}^n}{n!} \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\beta}}}\lambda_{\acute{\beta}}^n}{n!} \Big[\sum_i^{n_{\kappa\beta}} Q_i^\kappa + \sum_j^{n_{\acute{\kappa}}} Q_j^\kappa \Big] \Big[\sum_i^{n_{\kappa\beta}} Q_i^\beta + \sum_j^{n_{\acute{\beta}}} Q_j^\beta \Big] \Big\rangle \tag{13}$$

The first term of (13) will be the product of sums over coincident events. This is the sum all the elements of a matrix which are of products of the individual depositions. The product of these two sums will include two types of terms - those that correspond to energies all deposited into the same element, and those that contain energies from two separate elements. Taking advantage of the fact that we are calculating the expectation value over an entire pulse (or an arbitrary set of N_{mc} events), which will contain a large number of depositions, we can separate this sum into diagonal and off-diagonal parts and reexpress them in terms of the basic averages:

$$\Big\langle \sum_i^{n_{\kappa\beta}} Q_i^\kappa \sum_j^{n_{\kappa\beta}} Q_j^\beta \Big\rangle = \Big\langle \sum_{i=j}^{n_{\kappa\beta}} Q_i^\kappa Q_j^\beta + \sum_{i\neq j}^{n_{\kappa\beta}} Q_i^\kappa \sum_j^{n_{\kappa\beta}} Q_j^\beta \Big\rangle = \Big\langle \sum_{i=j}^{n_{\kappa\beta}} Q_i^\kappa Q_j^\beta \Big\rangle + \Big\langle \sum_{i\neq j}^{n_{\kappa\beta}} \sum_j^{n_{\kappa\beta}} Q_i^\kappa Q_j^\beta \Big\rangle$$

We see that there are n diagonal terms, and $n^2 - 1$ off-diagonal ones. Now we take advantage of the expectation value operator and write both parts in terms of the number of events and the pulse average of the basic quantities:

$$\left\langle \sum_i^{n_{\kappa\beta}} Q_i^\kappa \sum_j^{n_{\kappa\beta}} Q_j^\beta \right\rangle = n\widehat{Q^\kappa Q^\beta} + (n^2 - n)\widehat{Q^\kappa}\widehat{Q^\beta} \tag{14}$$

In the off-diagonal sum, we can express the average of the products as a product of the averages because events which deposit energy into different elements during different events $(i \neq j)$ will not be correlated. We can take the Poisson factors out of the expectation value, and now calculate the deposition sums using our notation for the different averages:

$$\begin{aligned}\langle E^\kappa E^\beta \rangle = &\sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!}\left[n\widehat{Q^\kappa Q^\beta} + (n^2 - n)\widehat{Q}^\kappa \widehat{Q}^\beta\right] + \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\kappa}}}\lambda_{\acute{\kappa}}^n}{n!} n\acute{Q}^\kappa \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\beta}}}\lambda_{\acute{\beta}}^n}{n!} n\acute{Q}^\beta \\ &+ \sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!} n\widehat{Q}^\beta \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\kappa}}}\lambda_{\acute{\kappa}}^n}{n!} n\acute{Q}^\kappa + \sum_n^{N_{mc}} \frac{e^{-\lambda_{\kappa\beta}}\lambda_{\kappa\beta}^n}{n!} n\widehat{Q}^\kappa \sum_n^{N_{mc}} \frac{e^{-\lambda_{\acute{\beta}}}\lambda_{\acute{\beta}}^n}{n!} n\acute{Q}^\beta \end{aligned} \tag{15}$$

To evaluate (15), we need

$$\sum_n^{N_{mc}} \frac{e^{-\lambda_\kappa}\lambda_\kappa^n n^2}{n!} = \bar{n}_\kappa^2 + \bar{n}_\kappa$$

Note that each element pair produces a unique $\lambda_{\kappa\beta} = \widehat{n}_{\kappa\beta}$. Now we can calculate the Poisson sums:

$$= \widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta} + \widehat{n}_{\kappa\beta}^2 \widehat{Q}^\kappa \widehat{Q}^\beta + \acute{n}_\kappa \acute{Q}^\kappa \acute{n}_\beta \acute{Q}^\beta + \widehat{n}_{\kappa\beta}\widehat{Q^\beta}\acute{n}_\kappa \acute{Q}^\kappa + \widehat{n}_{\kappa\beta}\widehat{Q^\kappa}\acute{n}_\beta \acute{Q}^\beta$$

$$= \widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta} + \left(\widehat{n}_{\kappa\beta}\widehat{Q^\kappa} + \acute{n}_\kappa \acute{Q}^\kappa\right)\left(\widehat{n}_{\kappa\beta}\widehat{Q^\beta} + \acute{n}_\beta \acute{Q}^\beta\right) \tag{16}$$

Recall from (11) that the sum over coincident and unique events for an element is equal to the total sum:

$$\langle E^\kappa E^\beta \rangle = \widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta} + \bar{n}_\kappa \overline{Q^\kappa}\bar{n}_\beta \overline{Q^\beta} \tag{17}$$

Substitute this into (9) to get an expression for the signal covariance:

$$\sigma_{Y^\kappa Y^\beta} = \widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta} + \bar{n}_\kappa \overline{Q^\kappa}\bar{n}_\beta \overline{Q^\beta} - \bar{n}_\kappa \overline{Q^\kappa}\bar{n}_\beta \overline{Q^\beta} = \widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta} \tag{18}$$

Now we have an expression for the asymmetry covariance:

$$\sigma_{\alpha_\kappa \alpha_\beta} = \frac{\widehat{n}_{\kappa\beta}\widehat{Q^\kappa Q^\beta}}{2G_\kappa G_\beta \bar{n}_\kappa \overline{Q^\kappa}\bar{n}_\beta \overline{Q^\beta}} \tag{19}$$

Finally, we invert the matrix of uncertainty covariances and sum the elements to calculate the physics asymmetry:

$$\frac{1}{\sigma_\alpha^2} = \sum_\kappa^K \sum_\beta^K [\sigma_{\alpha_\kappa \alpha_\beta}]_{\kappa\beta}^{-1} \tag{20}$$

4 Diagonal Approximation

We can approximate the uncertainty in the asymmetry by summing only the inverse of the diagonal terms of the uncertainty covariance matrix. This is equivalent to ignoring covariances and only calculating element variances. This approximation is valid when $l/a < 1$, where l is the length that reaction products travel, and a is the dimension of the cells.

$$\sum_{\kappa}^{K}\sum_{\beta}^{K}[\sigma_{\alpha_\kappa\alpha_\beta}]^{-1}_{\kappa\beta} \approx \sum_{\kappa}^{K}\frac{1}{\sigma^2_{\alpha_\kappa}} \tag{21}$$

Consider the variances for the asymmetry of each element

$$\sigma^2_{\alpha_\kappa} = \frac{\langle E^{\kappa 2}\rangle - \langle E^\kappa\rangle^2}{2G_\kappa^2\bar{n}_\kappa^2\overline{Q^\kappa}^2} \tag{22}$$

To evaluate this, we must calculate the expectation value of the diagonal elements of the signal covariance matrix:

$$\langle E^{\kappa 2}\rangle = \sum_{n}^{N_{mc}}\frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!}\langle\sum_{i=1}^{n}Q_i^\kappa\rangle^2 = \sum_{n}^{N_{mc}}\frac{e^{-\lambda_\kappa}\lambda_\kappa^n}{n!}\left[n\overline{Q^{\kappa 2}} + (n^2-n)\overline{Q^\kappa}^2\right] \tag{23}$$

Now we can simplify the formula for the variance $\sigma^2_{\alpha_\kappa}$:

$$\sigma^2_{\alpha_\kappa} = \frac{\bar{n}_\kappa\overline{Q^{\kappa 2}} + (\bar{n}_\kappa^2+\bar{n}_\kappa)\overline{Q^\kappa}^2 - \bar{n}_\kappa\overline{Q^\kappa}^2 - \bar{n}_\kappa^2\overline{Q^\kappa}^2}{2\bar{n}_\kappa^2G_\kappa^2\overline{Q^\kappa}^2} = \frac{1}{2\bar{n}_\kappa}\frac{\overline{Q^{\kappa 2}}}{G_\kappa^2\overline{Q^\kappa}^2} \tag{24}$$

This solution corresponds to the simplification of (19) with repeated indices. This approximation can be used as a check on the covariant solution in the appropriate regime.

5 Additional Computed Quantities

Aside from basic quantities such as (6), and (17), we will also calculate some additional factors. Define the efficiency of element κ, ϵ_κ, as:

$$\epsilon_\kappa = \frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}\delta_i^\kappa,\ \ where\ \ \delta_i^\kappa = \left\{\begin{array}{ll} 0, & \text{if } Q_i^\kappa = 0 \\ 1, & \text{if } Q_i^\kappa \neq 0 \end{array}\right\} \tag{25}$$

The quantitity ϵ_κ should converge, so the expected number of deposition events in element κ, n^κ_{exp}, can be predicted by multiplying the element efficiency by the number of experimental neutron events, N_{exp}:

$$n^\kappa_{exp} = \epsilon_\kappa N_{exp}$$

Define the multiplicity, ν, and energy efficiency η, of the chamber as:

$$\nu = \frac{1}{N_{mc}}\sum_{i}^{N_{mc}}\sum_{\kappa}^{K}\delta_i^\kappa = \sum_{\kappa}^{K}\epsilon_\kappa; \tag{26}$$

$$\eta = \frac{1}{N_{mc}\ Q^{n(^3He,T)p}}\sum_{i}^{N_{mc}}\sum_{\kappa}^{K}Q_i^\kappa \tag{27}$$