University of Kentucky, Physics 306
Homework #2, Rev. A, due Monday, 2022-01-31

1. Conformal maps: for each of the functions $w = f(z)$ mapping $z = x + iy \mapsto w = u + iv$, plot: i) the contours of $u(x, y)$ and $v(x, y)$ in the z-plane; and ii) the two families of curves $f(x + iy)$, parameterized by constant x or y respectively, in the w-plane.

a) $f(z) = z + c$ for a complex constant $c = a + ib$ (use $c = 2 + i$).

b) $f(z) = cz$ for the same constant c.

c) $f(z) = z^2$. Show the two branches in the z-plane and the corresponding branch cut in w.

d) $f(z) = e^z$, the transformation to polar coordinates $z = \rho e^{i\phi}$. [bonus: $f(z) = \sin(z)$].

e) $f(z) = e^z + z$. A contour of this function was used by Rogowski to create a smooth edge of and electrode without any "hot spots" of high electric field which would arc. The other family of contours represents the lines of electric flux and corresponding charge on the electrode.

2. Projections: relational versus parametric linear/planar geometry.

a) Show graphically that the following equations define the set of points $\{\boldsymbol{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\{\boldsymbol{x} \mid \boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}\}$	$\{\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha \mid \alpha \in \mathbb{R}\}$
plane	$\{\boldsymbol{x} \mid \boldsymbol{A} \cdot \boldsymbol{x} = D\}$	$\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma \mid \beta, \gamma \in \mathbb{R}\}$

where $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$ are constants which define the geometry, $\boldsymbol{x}_{1,2}$ are fixed points on the line and plane respectively, and α, β, γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane). [bonus: Show the 5th relation $\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{A} \times \boldsymbol{\delta} \mid \boldsymbol{\delta} \in \mathbb{R}^3\}$.]

b) What constraint between $\boldsymbol{a}$ and $\boldsymbol{d}$ is implicit in the formula $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$? What is the relation between $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{A}$? Substitute $\boldsymbol{x}$ of each parametrization into its relational equation to show the consistency between the two forms and to derive $\boldsymbol{d}$ and D in terms of $\boldsymbol{a}, \boldsymbol{x}_1$ and $\boldsymbol{A}, \boldsymbol{x}_2$.

c) Define $\tilde{\boldsymbol{a}} \equiv \boldsymbol{A}/(\boldsymbol{a} \cdot \boldsymbol{A})$, which is parallel to $\boldsymbol{A}$, and *normalized* in the sense that $\boldsymbol{a} \cdot \tilde{\boldsymbol{a}} = 1$. Using the BAC-CAB rule, show that $\boldsymbol{x} = \boldsymbol{a}(\tilde{\boldsymbol{a}} \cdot \boldsymbol{x}) - \tilde{\boldsymbol{a}} \times (\boldsymbol{a} \times \boldsymbol{x})$ for all $\boldsymbol{x}$. Illustrate this non-orthogonal projection of $\boldsymbol{x}$ onto vectors $\boldsymbol{x}_1$ parallel to the line plus $\boldsymbol{x}_2$ parallel to the plane. [bonus: use this to calculate the point $\boldsymbol{x}_0$ at the intersection of the line and plane in terms of $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$. Verify this by showing that $\boldsymbol{x}_0$ satisfies the relational equation for both the line and plane.]

d) Let $\tilde{\boldsymbol{a}} \equiv \frac{\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, $\tilde{\boldsymbol{b}} \equiv \frac{\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, and $\tilde{\boldsymbol{c}} \equiv \frac{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, where the arrows on $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ distinguish them from their *covectors* $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}$. The definition of $\tilde{\boldsymbol{a}}$ is the same as in c) with $\boldsymbol{A} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}$. Calculate the nine combinations of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})^T \cdot (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}) = I$, i.e. $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{a}} = 1$, $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{b}} = 0$, ... to show they are *mutually orthonormal*. In this sense, $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the dual basis of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$. [bonus: Show using Cramer's rule that $[\tilde{\boldsymbol{a}}|\tilde{\boldsymbol{b}}|\tilde{\boldsymbol{c}}] = [\vec{\boldsymbol{a}}|\vec{\boldsymbol{b}}|\vec{\boldsymbol{c}}]^{-1}$, ie. that it is a *reciprocal basis*.]

e) The *contravariant components* of $\boldsymbol{x}$ are defined as the components (α, β, γ) that satisfy the equation $\boldsymbol{x} = \vec{\boldsymbol{a}}\alpha + \vec{\boldsymbol{b}}\beta + \vec{\boldsymbol{c}}\gamma$; i.e., $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$ is the contravariant basis. Using $\boldsymbol{x} \cdot \tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\boldsymbol{x}$ in terms of dot products. Likewise, find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of $\boldsymbol{x}$, defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy $\boldsymbol{x} = \tilde{\boldsymbol{a}}\tilde{\alpha} + \tilde{\boldsymbol{b}}\tilde{\beta} + \tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the covariant basis.