University of Kentucky, Physics 306
Homework #3, Rev. B, due Monday, 2022-02-07

0. The **Pauli matrices**, defined as $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, are used to describe quantum mechanical spin-1/2 particles. Show that $\sigma_j \sigma_k = \delta_{jk} I + i\varepsilon_{jk\ell}\sigma_\ell$. What is the *triple product* $\sigma_x\sigma_y\sigma_z$? Thus σ_j acts like a basis vector, the product of the Pauli matrices has the structure of the dot, cross, and triple products.

[bonus: Show that product of *Dirac matrices* is $\gamma^\mu\gamma^\nu = g^{\mu\nu} I + i\sigma^{\mu\nu}$ and calculate the components $g^{\mu\nu}$ and the matrices $\sigma^{\mu\nu}$.]

1. Vector rotations can be generated by the matrix $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the same way complex rotations $e^{i\phi}$ are generated by i with the property $i^2 = -1$, representing a 90° rotation.

a) Show that $M\boldsymbol{v}$ also rotates the vector $\boldsymbol{v} = (\, x \quad y \,)^T$ by 90° CCW, and that $M^2 = -I$.

b) Show that $R = e^{M\phi} = I\cos\phi + M\sin\phi$ and calculate the components of R. Show that $R^T R = I$ and that $dR = MRd\phi$, similar to $(e^{i\phi})^*(e^{i\phi}) = 1$ and $de^{i\phi} = ie^{i\phi}d\phi$.

c) The cross product $\times$ also generates rotation. Calculate the matrices $\boldsymbol{M} = (M_x, M_y, M_z)$, where $\hat{\boldsymbol{x}} \times \boldsymbol{v} = M_x\boldsymbol{v}$, etc. Show that the components of M_ℓ are $(M_\ell)_{jk} = \varepsilon_{kj\ell}$ and that $M_\ell^2 = -P_{\perp\ell}$, where $P_{\perp\ell}$ projects perpendicular to $\hat{\boldsymbol{e}}_\ell$. Thus the general rotation matrix $R_{\boldsymbol{v}}$ for a CCW rotation by an angle $v = |\boldsymbol{v}|$ about the $\hat{\boldsymbol{v}}$-axis is given by the *Rodrigues' formula* $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}} = I\cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}}\sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1-\cos v)$. The third term projects out the axis of rotation $\hat{\boldsymbol{v}}$.

2. Minkowski space: We will derive the Lorentz transformations using matrices from Einstein's principle of special relativity that the speed of light c is constant in any reference frame.

a) Let $\Delta x = c\Delta t$ for the distance Δx travelled by a photon in time Δt. Combining them into a *space-time vector* $\boldsymbol{x} = (\, c\Delta t \quad \Delta x \,)^T$, show that this is equivalent to $\boldsymbol{x}^2 = \boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{x}^T g\boldsymbol{x} = 0$, where the matrix $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the *Minkowski metric*. In three dimensional space the metric is a 4×4 matrix and x^2 becomes $r^2 = x^2 + y^2 + z^2$.

b) Momentum and energy can also be combined into a space-time vector $\boldsymbol{p} = (\, E/c \quad p \,)^T$. Show that invariance of the dot product $\boldsymbol{p}^T g\boldsymbol{p} = -(mc)^2$ leads to the formula $E^2 = (pc)^2 + (mc^2)^2$, which reduces to $E = mc^2$ when $p = 0$.

c) Normal rotations R keep the length of vector constant—they preserve Euclidian metric I, or $R^T I R = I$. Likewise, Lorentz transformations Λ preserve the Minkowski metric: $\Lambda^T g\Lambda = g$. For a small 'rotation' $\Lambda = I + G d\alpha$ generated by G, show that $G^T g + gG = 0$. Show that $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies this relation. Note the difference between G and M!

d) Show that $G^2 = I$ and therefore $\Lambda = e^{G\alpha} = I\cosh\alpha + G\sinh\alpha = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$, where $\beta = \tanh\alpha = v/c$ and $\gamma = \cosh\alpha = (1-\beta^2)^{-1/2}$. Thus the Lorentz transformations $\boldsymbol{x}' = \Lambda\boldsymbol{x}$ are $t' = \gamma(t + vx/c^2)$ and $x' = \gamma(x + vt)$, which encode all features of special relativity.