University of Kentucky, Physics 306
Homework #4, Rev. B, due Monday, 2022-02-14

1. Similarity transforms. The Pauli matrices $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ describe the same operator in different coordinates, just as $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ point in different directions.

a) Show that $U_x^\dagger U_x = U_y^\dagger U_y = I$ where $U_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $U_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. Thus the columns of U_x and U_x are unit vectors and U_i are unitary transformations under the cannonical metric.

b) Show that $\sigma_x U_x = U_x \sigma_z$ and $\sigma_y U_y = U_y \sigma_z$. Thus the matrices of the operators σ_x and σ_y are both σ_z in the new bases U_x, U_y respectively. For $i, j = x, y, z$, find the matrix U_{ij} that gives the *similarity transform* $\sigma_i = U_{ij}\sigma_j U_{ij}^\dagger$. Note that $_{ij}$ indexes the matrix U, not its components.

c) The *trace* of a matrix is the sum of its diagonal elements: $\text{tr}A = A_{ii}$ in index notation. It has the geometric interpretation as the 'perimeter' of a transformation, which should not depend on the basis. Calculate tr(0) (matrix of zeros), and show that $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$ but not necessarily $\text{tr}(ABC) = \text{tr}(CBA)$. Thus show that if $A' = UAU^\dagger$, then $\text{tr}(A') = \text{tr}(A)$.

d) The *determinant* of a matrix is the fully antisymmetric product of one element in each row or column: $\det A = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$ in index notation. It has the geometric interpretation as the 'volume' of a transformation, which should not depend on the basis either. Calculate $\det(I)$, and show that $\det(U) = 1$ if $U^\dagger U = I$, given that $\det(AB) = \det(A)\det(B)$. Thus show that if $A' = UAU^\dagger$, then $\det(A') = \det(A)$. [bonus: show that $\exp(\text{tr}(A)) = \det(\exp(A))$, for example, $\exp(\text{tr}(0)) = \det(\exp(0) = I)$.]

2. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed $A = RS$ into a stretch S and a rotation R, which are the building blocks of all linear operators. This problem explores the structure of stretches.

a) Use the exponential form $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}}$ and the asymmetry of the generator $M^\dagger = -M$ to show that $R^\dagger R = I$ or $R^\dagger = R^{-1}$. In contrast, a stretch is symmetric or *Hermitian* $S^\dagger = S$.

b) A symmetric matrix S is guaranteed to have a complete set of eigenvectors $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, such that $S\boldsymbol{v}_i = \boldsymbol{v}_i\lambda_i$. Show that we can extend this equation to $SV = VW$, where $W = \text{diag}(\lambda_1, \lambda_2, \ldots)$ is *diagonal*.

c) Show that two eigenvectors $\boldsymbol{v}_i, \boldsymbol{v}_j$ of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$, and therefore the matrix of eigenvectors is an orthogonal transformation: $V^\dagger V = I$. [bonus: Interpret the resulting decompositions $S = VWV^\dagger$ and $W = V^\dagger SV$ in terms of rotations.]

d) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Do they look familiar? Show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify H03#1b). Hermitian matrices have real eigenvalues while anti-Hermitian matrices have zero trace and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite and has real positive eigenvalues, while the exponential of an anti-Hermitian matrix is unitary with unit modulus determinant and eigenvalues. This is the normal matrix analogy, which completes the connection between matrices and complex numbers.

3. The **Cayley-Hamilton Theorem** states that any matrix A satisfies its *characteristic equation* $|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, substituting $\lambda \to A$.

a) [bonus: Show that the characteristic equation is invariant under similarity transforms. Thus an $n \times n$ matrix has n independent invariants: either the eigenvalues of A or the coefficients of its characteristic equation. The first and last coefficients of this equation are the determiant and trace, which where already shown above to be invariant. The other coefficients represent k-dimensional 'perimeters', for example, the surface area, of the transformation.]

b) [bonus: Prove the Cayley-Hamilton Theorem for diagonal matrices. The theorem then follows for all diagonalizable matrices by invariance of the characteristic equation.]

c) [bonus: Show that $A^{-1} = -(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I)/a_0$.]