University of Kentucky, Physics 306
Homework #5, Rev. B, due Monday, 2022-02-21

1. The good, ...

a) The *commutator* $[A, B]$ of two matrices A and B is defined as the matrix $[A, B] \equiv AB - BA$. Calculate $[\sigma_j, \sigma_k]$ for the *Pauli matrices* σ_i, $i = x, y, z$. Do they commute? Compare with H03#0: $\sigma_j \sigma_k = I\sigma_j \cdot \sigma_k + i\sigma_j \times \sigma_k$, treating the Pauli matrices $\sigma_k \sim \hat{\boldsymbol{e}}_k$ as unit vectors.

b) Show that the commutator of A and B is antisymmetric: $[A, B] = -[B, A]$, and bilinear: $[\alpha_i A_i, B_j \beta_j] = \alpha_i [A_i, B_j]\beta_j$. Thus it is a 'matrix version' of the vector cross product, so $[A, A] = 0$, and two matrices which commute are in the 'same direction' (both are diagonal in some basis).

c) Verify that the two matrices $A = \begin{pmatrix} 9 & -2 & -6 \\ -2 & 9 & -6 \\ -6 & -6 & -7 \end{pmatrix}$ and $B = \begin{pmatrix} 54 & 10 & -3 \\ 10 & -45 & 30 \\ -3 & 30 & 46 \end{pmatrix}$.
commute. Simultaneously diagonalize both matrices. Are the eigenvectors orthogonal?
Hint: Octave or Mathematica is your friend!

d) By adding and subtracting $M = H + iK$ and its adjoint $M^\dagger = H - iK$, where $H^\dagger = H$ and $K^\dagger = K$ are Hermitian (note that $A = iK$ is antisymmetric: $A^\dagger = -A$), show that any matrix M can be decomposed into its symmetric and antisymmetric parts, similar to how $z = x + iy$ projects into its real and imaginary parts.

e) A *normal* matrix is one that commutes with its adjoint: $[N, N^\dagger] = 0$. Show that $[N, N^\dagger] = 2i[H, K]$, where $N = H + iK$. This shows that a normal matrix is one whose symmetric and antisymmetric parts commute. Thus normal matrices are the most general matrices that can be diagonalized with orthogonal eigenvectors. The eigenvalues of N are $n_j = h_j + ik_j$ where h_j and k_j are the eigenvalues of H and K. Thus N acts like independent complex numbers!

f) [bonus: In analogy with polar complex numbers $z = e^{\sigma+i\phi} = \rho u$, where $\rho = e^\sigma > 0$ and $u = e^{i\phi}$ is a *unit*: $|u|^2 = u^* u = 1$, any matrix A has a *polar decomposition* $A = RS$ into a rotation R and a stretch S, which are the building blocks of all linear operators. By taking the exponential $M = e^N$ of $N = H + iK$, show that any normal matrix can be decomposed into $M = UP$, where U is unitary ($U^\dagger U = I$) and P is positive definite ($P^\dagger = P$ with non-negative eigenvalues).]

g) [bonus: Using the polar decomposition $A = RS$, where $R^\dagger R = I$ and $S^\dagger = S$, and diagonalizing $S = VWV^\dagger$, show that any matrix may be decomposed $A = UWV^\dagger$, where $U = RV$ and V are unitary, and W is diagonal. This *singular value decomposition* of matrices is the generalization the spectral theorem for all matrices. Because it involves both stretches and rotations, every *singular value* (eigenvalue) w_i has two eigenvectors: a *left singular vector* $\vec{u}_i$ and a *right singular vector* $\vec{v}_i$. This is useful for transformations from one vector space into another, not just operators.]

2. The bad, . . .

a) The Pauli *ladder operators* $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ transform the unit vectors $(1, 0)^T$ and $(0, 1)^T$ from one to the next along a chain, *annihilating* the last one. Calculate $\sigma_\pm$, $\sigma_\pm^2$, $\sigma_\pm^3, \ldots, \sigma^n$. How does this generalize to higher dimension?

b)Calculate the doubly *degenerate* eigenvalue λ of $\sigma_\pm$ and show that they have only one eigenvector $\vec{v}_1$. The other vectors associated with λ are called *generalized eigenvectors*, defined by $M\vec{v}_k = \vec{v}_k \lambda_k + \vec{v}_{k-1}$. Note that $\vec{v}_1$ follows the pattern if we define $\vec{v}_0 = \vec{0}$. Find the matrix of M in the basis $\{\vec{v}_k\}$ (called a Jordan block).

c) Calculate the adjoint $\sigma_{\pm}^{\dagger}$, the *projections* $P_+ = \sigma_+\sigma_-$ and $P_- = \sigma_-\sigma_+$, and the commutators $[\sigma_+, \sigma_-]$, $[I, \sigma_{\pm}]$, and $[\sigma_z, \sigma_{\pm}]$. Note that $\sigma_{\pm}$ commutes with I because its eigenvalues are all the same, but not with σ_z with distinct eigenvalues. This is true in general. [bonus: use the last commutation relation to show that if $\vec{v}$ is an eigenvector of σ_z, the $\sigma_{\pm}\vec{v}$ is also an eigenvector. Thus $\sigma_{\pm}$ raise an lower one one eigenvector of σ_z into another, like climbing up or down the rungs of a ladder, which is where their name comes from.]

d) Show that $\sigma_x\sigma_{\pm}\sigma_x^{-1} = \sigma_{\mp}$ Find the most general 2×2 *nilpotent* operator via the similarity transform $V\sigma_+V^{-1}$ of σ_+ into a generic basis $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Represent it as an outer product. What is its geometrical interpretation? Do the same for a generic *diagonalizable* operator to represent it as the sum of two outer products (its *spectral decomposition*).

e) Find the matrix of the derivative operator $D = d/dx$ in the linear space of polynomials $A(x) = a_0 + a_1x + a_xx^2 + \ldots + a_nx^n$, using the monomials $\hat{\boldsymbol{e}}_k = x^k$ as basis functions. Show that $D^n = 0$ and, thus the derivative is nilpotent.