University of Kentucky, Physics 306
Homework #6, Rev. B, due Monday, 2022-02-28

1. The **Fourier Series** is an application of using an orthogonal basis to extract components of a generalized vector (function). Orthogonality is cruitial, because otherwise it would be impossible to solve the linear system of equations (invert an infinite-dimesional matrix) to obtain each component.

a) Submit a screenshot of your completed Level 3 in the Fourier: Making Waves Wave Game.

b) Calculate the Fourier cosine series $f(x) = \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$ of $f(x) = kx^2$ over the interval $-L/2 \leq x \leq L/2$. What does the periodic series look like outside this interval? Compare the inner product $\langle kx^2|kx^2\rangle = \int_{-L/2}^{L/2} dx\ \left(kx^2\right)^2$ with that of its Fourier components $\sum_n a_n^2$. [it's a rotation!]

2. The **Legendre Polynomials** $P_\ell(x)$ appear in the solution of Laplace's equation in spherical coordinates. They can be defined as *the* series of orthogonal polynomials with respect to weight $w(x) = 1$ over the range $-1 \leq x \leq 1$, of degree $\ell = 0, 1, 2, \ldots$, standardized to $P_\ell(1) = 1$.

a) Show that $\langle x^m|x^n\rangle = \frac{2}{m+n+1}$ [$m+n$ is even] or 0 [$m+n$ is odd]. Apply the Gram-Schmidt procedure to the basis of monomials $1, x, x^2, \ldots$ to generate the series of $P_\ell(x)$ up to order $\ell = 4$. Calculate the *nomalization* $h_\ell = \langle P_\ell|P_\ell\rangle$. Expand $f(x) = x^4$ in the Legendre basis.

b) Show that these basis functions $P_\ell(x)$ are eigenfunctions of the linear Lengendre differential operator $L = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ by direct calculation of their eigenvalues $L[P_\ell(x)] = \lambda P_\ell(x)$.

3. The **Sturm-Liouville** 2nd order linear differential operator

$$L[y(x)] \equiv \frac{1}{w(x)}\left[\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right]y(x) \tag{1}$$

is self-adjoint, $L^\dagger = L$, [it's a stretch!] with respect to the inner product

$$\langle y_1|y_2\rangle \equiv \int_a^b w(x)dx\ y_1(x)^*\, y_2(x) \tag{2}$$

if we impose the boundary conditions $y(a)w(a) = y(b)w(b) = 0$. Thus L has real eigenvalues λ_i and a complete set of orthogonal eigenfunctions. Any function $f(x)$ has the expansion $|f\rangle = \sum_i |u_i\rangle f_i$ or $f(x) = \sum_i u_i(x) f_i$, with components $f_i = \langle u_i|f\rangle/\langle u_i|u_i\rangle = \int_a^b w(x)dx\ u_i(x)^* f(x)/h_i$.

a) Show that L is self-adjoint or Hermitian. *Hint:* use the definition $\langle f|H^\dagger g\rangle \equiv \langle Hf|g\rangle$ to show that the operator $D = \frac{1}{w}\frac{d}{dx}$ is anti-Hermitian and apply it to the composition of operators in L. [bonus: Show that any linear 2^{nd}-order differential operator over any weight can be put in this form.]

b) Given eigenfunctions $L|u_i\rangle = \lambda_i|u_i\rangle$, show that $\lambda_i \in \mathbb{R}$ and that $\langle u_i|u_j\rangle = 0$ if $\lambda_i \neq \lambda_j$. Note that completeness is much harder to prove. Operate L on the expansion of $|f\rangle$ in the basis $|u_i\rangle$ to show that its *spectral decomposition* is $L = \sum_i \lambda_i |u_i\rangle\langle u_i|$. Do the same for the identity $I|f\rangle = |f\rangle$.

c) Compile a chart of $w(x), a, b, p(x), q(x), \lambda_i, h_i$ for each of these *orthogonal function bases*: i) cylindrical harmonics $e^{im\phi}$; ii) associated Legendre functions $P_\ell^{|m|}(x)$; iii) Fourier series $\sin(k_n x)$; iv) Bessel functions $J_m(k_n x)$; v) spherical Bessel functions $j_l(k_n x)$; vi) Hermite polynomials $H_n(x)$; vii) associated Laguere polynomials $L_n^{(\alpha)}(x)$. [see NIST Digital Library of Mathematical Functions]