

University of Kentucky, Physics 306
Homework #8, Rev. D, due Monday, 2022-04-04

1. The gradient, curl, and divergence are all applications of d in different dimensions. They are defined as $df = \nabla f \cdot d\mathbf{l}$, $d(\mathbf{A} \cdot d\mathbf{l}) = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$, $d(\mathbf{B} \cdot d\mathbf{a}) = (\nabla \cdot \mathbf{B}) \cdot d\tau$, respectively, in terms of the differential operator $d = dq^i \partial_i$ and elements $d\mathbf{l} = \hat{\mathbf{e}}_i h_i dq^i$, $d\mathbf{a} = \frac{1}{2} d\mathbf{l} \times d\mathbf{l}$, $d\tau = \frac{1}{3} d\mathbf{l} \cdot d\mathbf{a}$.

a) Apply these definitions to generic fields to obtain the formulas

$$\nabla f = \frac{\hat{\mathbf{e}}_i}{h_i} \frac{\partial}{\partial q^i} f, \quad \nabla \times \mathbf{A} = \frac{\hat{\mathbf{e}}_i}{h_j h_k} \frac{\partial}{\partial q^j} h_k A_k, \quad \nabla \cdot \mathbf{B} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q^k} h_i h_j B_k, \quad \text{for } i, j, k \text{ cyclic.}$$

b) Expand these formulas in Cartesian, cylindrical, and spherical coordinates and compare your results with [Wikipedia: Del in cylindrical and spherical coordinates](#).

c) Calculate the gradient of $f(r, \theta, \phi) = r^\ell P_\ell(\cos(\theta))$, using $\frac{x^2-1}{\ell} P'_\ell(x) = x P_\ell(x) - P_{\ell-1}(x)$.

d) Calculate the curl of $\mathbf{A}(x, y, z) = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)/(x^2 + y^2)^n$ in Cartesian and cylindrical coordinates.

e) Calculate the divergence of $\mathbf{B}(x, y, z) = (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)/(x^2 + y^2 + z^2)^n$ in Cartesian and spherical coordinates.

f) Derive the formulas for $d^2 f$ and $d^2 \mathbf{A} \cdot d\mathbf{l}$ in terms of their partial derivatives to show that the trivial 2nd derivatives $\nabla \times \nabla f$ and $\nabla \cdot \nabla \times \mathbf{A}$ are special cases of $d^2 = 0$.

2. Potential theory—the *Fundamental Theorem of Differentials (FTD)* $d \int_r \omega + \int_r d\omega = \omega$ implies that if $d\omega = 0$, then the *potential* $\alpha \equiv \int_r \omega \equiv [\int_{r'=0}^r \omega_{r\Omega}(r', \Omega) dr'] d\Omega$ (integrated along radial coordinate lines) is its *antiderivative*: $d\alpha = \omega$. We will use this formula to derive the antiderivative of the following special vector fields, generalizing the *Fundamental Theorem of Calculus (FTC)*:

a) Show that if $\nabla \times \mathbf{E} = \mathbf{0}$ (ie. \mathbf{E} is *irrotational*), then $\mathbf{E} = -\nabla V$ (ie. \mathbf{E} is *conservative*), with the potential function $V(\mathbf{r}) = -\int_r \mathbf{E} \cdot d\mathbf{l} + C$, where C is any constant. Show that for a radial path in spherical coordinates, this reduces to $V = -\mathbf{r} \cdot \int_0^1 \mathbf{E}(\lambda \mathbf{r}) d\lambda$.

b) Show that if $\nabla \cdot \mathbf{B} = 0$ (ie. \mathbf{B} is *incompressible*), then $\mathbf{B} = \nabla \times \mathbf{A}$ (ie. \mathbf{B} is *solenoidal*), with the potential function $\mathbf{A} \cdot d\mathbf{l} = \int_r \mathbf{B} \cdot d\mathbf{a} + d\chi$, where the ‘constant of integration’ $d\chi$ is the differential of any scalar field. Show that for a radial path in spherical coordinates, this reduces to $\mathbf{A} = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda$.

c) [bonus: Show that since the differential of any density field $\rho(\mathbf{r}) d\tau$ is zero, it can be written as the exact differential $\rho = \nabla \cdot (\mathbf{B} + \nabla \times \mathbf{A})$ of some flux field $\mathbf{B}(\mathbf{r}) \cdot d\mathbf{a}$, and any constant of integration (gauge) $\mathbf{A}(\mathbf{r}) \cdot d\mathbf{l}$. Derive the formula for \mathbf{B} for a radial path in spherical coordinates.]

3. Stokes’ theorems—the FTD also generalizes the FTC, integrating the derivative of a field over a region out to the boundary, resulting in an integral of the original field on the boundary.

a) Integrate $\int_S \nabla \times \mathbf{v} \cdot d\mathbf{a}$ where $\mathbf{v} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}xy$, and S is the parallelogram in the figure to the right. Integrate $\oint_{\partial S} \mathbf{v} \cdot d\mathbf{l}$ along the boundary ∂S of S to verify Stokes’ theorem.

b) Verify Stoke’s theorem with the function in #1d) on the disk $\rho < R, z = 0$ in the xy -plane centered at the origin.

c) Verify Gauss’ theorem with the function in #1e) inside the sphere $r < R$ of radius R centered at the origin.

