University of Kentucky, Physics 306
Homework #9, Rev. C, due Wednesday, 2022-04-13

1. Vector chain and product rules

a) Prove that i) $\nabla(f \circ g)(\boldsymbol{r}) = \nabla f(g(\boldsymbol{r})) = f'(g(\boldsymbol{r}))\nabla g(\boldsymbol{r})$; ii) $\nabla fg = g\nabla f + f\nabla g$; and iii) $\nabla \cdot f\boldsymbol{A} = \boldsymbol{A} \cdot \nabla f + f\nabla \cdot A$. [bonus: classify and prove all product rules]

b) Calculate i) ∇r^2, ii) ∇r, iii) $\nabla \times \boldsymbol{r}$, and iv) $\nabla \cdot \boldsymbol{r}$.

c) Use the results of a) and b) to calculate i) ∇r^m; ii) $\nabla\, \boldsymbol{k} \cdot \boldsymbol{r}$, iii) $\nabla \times (\boldsymbol{k} \times \hat{\boldsymbol{r}} r^m)$ (compare with H08#1d), iv) $\nabla \cdot (k\hat{\boldsymbol{r}} r^m)$, v) $\nabla \cdot \hat{\boldsymbol{r}}/r^n$ (compare with H08#1e), where $\boldsymbol{k}$ is a constant vector.

2. Scalar Laplacian and inverse: Green's function

a) Combine the formulas for divergence and gradient to obtain the formula for $\nabla^2 f(\boldsymbol{r})$, called the *scalar Laplacian*, in orthogonal curvilinear coordinates (q^1, q^2, q^3) with scale factors h_1, h_2, h_3.

b) Calculate $\nabla(1/r)$ and then $\nabla \cdot \nabla(1/r)$ using #1, and compare with $\nabla^2(1/r)$ from #2a). Where are the gradient and Laplacian singular?

c) Use Gauss' theorem to show that $\int_{r<a} \hat{\boldsymbol{r}}/r^2 = 4\pi$, and thus $-\nabla^2(1/4\pi r) = \delta^3(\boldsymbol{r})$. We call $G(\boldsymbol{\mathscr{r}}) \equiv -\nabla^{-2}\delta^3(\boldsymbol{\mathscr{r}}) = 1/4\pi\mathscr{r}$ the *Green's function.* $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$ shifts the *pole* from $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ to $\boldsymbol{r}'$.

d) Solve for a *particular solution* of the second order partial differential equation $-\nabla^2 V(\boldsymbol{r}) = \rho(\boldsymbol{r})$ by expanding $\rho(\boldsymbol{r}) = \int d^3\boldsymbol{r}' \rho(\boldsymbol{r}')\delta^3(\boldsymbol{\mathscr{r}})$, and applying c) to each basis function $\delta^3(\boldsymbol{\mathscr{r}}) = \delta^3(\boldsymbol{r} - \boldsymbol{r}')$. Thus the Green's function is use to invert the Laplacian operator!

3. Vector Laplacian and decomposition: Helmholtz theorem

a) Write down all possible combinations of gradient, curl, and divergence to form second vector derivatives of both scalar and vector fields. Which 'natural' second derivatives are zero? Show that $\nabla^2 \boldsymbol{F} = \nabla\nabla \cdot \boldsymbol{F} - \nabla \times (\nabla \times \boldsymbol{F})$, where $\nabla^2 = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$ (in Cartesian coordinates only) is called the *vector Laplacian.* Thus the Laplacian with its *longitudinal* $\nabla\nabla \cdot \boldsymbol{F}$ and *transverse* $\nabla \times (\nabla \times \boldsymbol{F})$ components is *the* unique second derivative of a vector field. In other coordinates, the Laplacian is defined in terms of curvilinear gradient, divergence and curls by the above equation. [bonus: expand the vector Laplacian in terms of scale factors in an orthogonal curvilinear coordinate system]

b) Use the technique of #2d) to solve the above equation for $\boldsymbol{F}$, and thus prove that any vector field $\boldsymbol{F}(\boldsymbol{r})$ can be decomposed into $\boldsymbol{F} = \boldsymbol{E} + \boldsymbol{B}$, an *irrotational* field $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ (the *longitudinal* part), and an *incompressible* field $\nabla \cdot \boldsymbol{B} = 0$, (the *transverse* part) with *sources* $\rho = \nabla \cdot \boldsymbol{E}$ and $\boldsymbol{J} = \nabla \times \boldsymbol{B}$, respectively. Show that the vector source must satisfy $\nabla \cdot \boldsymbol{J} = \boldsymbol{0}$. Show that $\boldsymbol{E} = -\nabla V$ (it is *conservative*) with the potential $V = -\nabla^{-2}\rho$ and that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ (it is *solenoidal*) with potential $\boldsymbol{A} = -\nabla^{-2}\boldsymbol{J}$. Thus any vector field $\boldsymbol{F}$ with sources that vanishes fast enough as $\boldsymbol{r} \to \infty$ is completely determined by its two sources $\rho = \nabla \cdot \boldsymbol{F}$ and $\boldsymbol{J} = \nabla \times \boldsymbol{F}$.