

L05 - Section 1.1 - Vector Algebra

* Linear (vector) space

~ linear combination: $(\alpha \vec{u} + \beta \vec{v})$ is the basic operation

~ basis: $(\hat{x}, \hat{y}, \hat{z})$ or $\vec{a}, \vec{b}, \vec{c}$ # basis elements = dimension

independence: not collapsed into lower dimension

closure: vectors span the entire space

~ components: $\vec{X} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma = (\vec{a} \ \vec{b} \ \vec{c}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

in matrix form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{where}$$

where

$$\vec{a} = \hat{x}a_x + \hat{y}a_y + \hat{z}a_z = (\hat{x} \ \hat{y} \ \hat{z}) \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \text{ etc}$$

(usually one upper, one lower index)

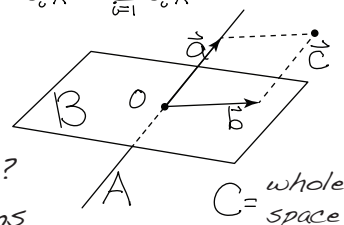
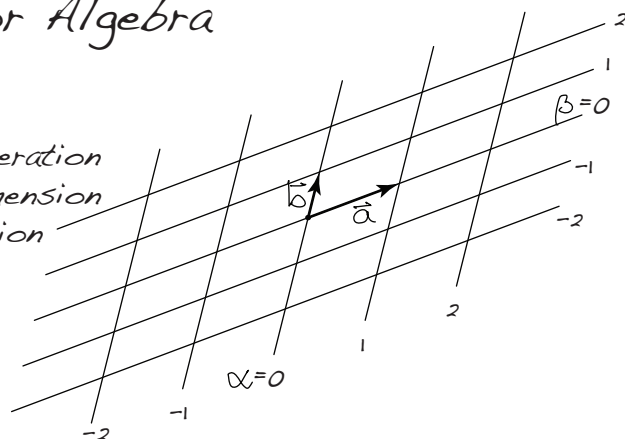
~ Einstein notation: implicit summation over repeated indices

$$\vec{X} = \vec{b}_i x^i = \sum_{i=1}^3 \vec{b}_i x^i$$

~ direct sum: $C = A \oplus B$ add one vector from each independent space to get vector in the product space (not simply union)

~ projection: the vector $\vec{c} = \vec{a} + \vec{b}$ has a unique decomposition ('coordinates' (\vec{a}, \vec{b}) in A, B) - relation to basis/components?

~ all other structure is added on as multilinear (tensor) extensions



* Metric (inner, dot product) - distance and angle

$$c = \vec{a} \cdot \vec{b} = ab \cos \theta = a_{||} b = a b_{||} = a_x b_x + a_y b_y + a_z b_z = a_i b^i = (a_x \ a_y \ a_z) \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

~ properties: 1) scalar valued - what is outer product?

2) bilinear form $a \cdot (b+c) = a \cdot b + a \cdot c$ $(a+b) \cdot c = a \cdot c + b \cdot c$

3) symmetric $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

~ orthonormality and completeness - two fundamental identities help to calculate components, implicitly in above formulas

$$\begin{aligned} \hat{e}_i \cdot \hat{e}_j &= \delta_{ij} \\ \sum_{i=1}^3 \hat{e}_i \hat{e}_i &= I \end{aligned}$$

Kronecker delta: components of the identity matrix

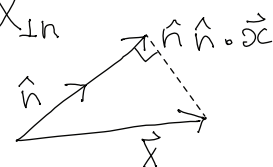
$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$a^i = \vec{a} \cdot \hat{e}_i = a^1 \hat{e}_1 \cdot \hat{e}_i + a^2 \hat{e}_2 \cdot \hat{e}_i + a^3 \hat{e}_3 \cdot \hat{e}_i$$

~ orthogonal projection: a vector \vec{n} divides the space X into $X_{||n} \oplus X_{\perp n}$

geometric view: dot product $\hat{n} \cdot \vec{x}$ is length of \vec{x} along \hat{n}

Projection operator: $P_{||} \equiv \hat{n} \hat{n} \cdot$ acts on x : $P_{||} \vec{x} = \vec{x}_{||} = \hat{n} \hat{n} \cdot \vec{x}$



~ generalized metric: for basis vectors which are not orthonormal, collect all $n \times n$ dot products into a symmetric matrix (metric tensor)

$$\begin{aligned} g_{ij} &= \vec{b}_i \cdot \vec{b}_j \\ \vec{x} \cdot \vec{y} &= x^i \vec{b}_i \cdot \vec{b}_j y^j = x^i g_{ij} y^j \\ &= \vec{x}^T \vec{b} \cdot \vec{b} \vec{y} = \vec{x}^T \mathbf{g} \vec{y} = \begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \end{aligned}$$

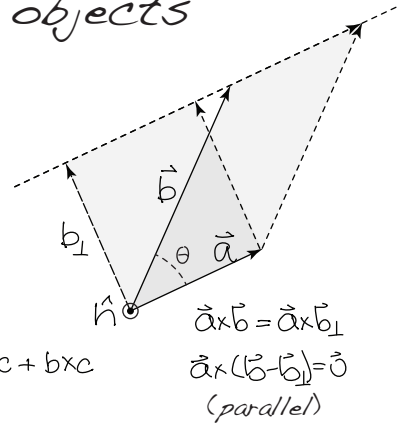
in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)

Exterior Products - higher-dimensional objects

* cross product (area)

$$\vec{c} = \vec{a} \times \vec{b} = \hat{n} a b \sin \theta = \hat{n} a_{\perp} b = \hat{n} a b_{\perp} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{n} \perp \vec{a}$ and $\hat{n} \perp \vec{b}$ (RHR-rule)



~ properties: 1) vector-valued

2) bilinear

$$a \times (b + c) = a \times b + a \times c \quad (a + b) \times c = a \times c + b \times c$$

3) antisymmetric $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (oriented)

$$\vec{a} \times (\vec{b} - \vec{b}_{\perp}) = \vec{0} \quad (\text{parallel})$$

~ components: $\hat{e}_i \times \hat{e}_j = \epsilon_{ij}^k \hat{e}_k$

$$\text{where } \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$$

Levi-Civita tensor - completely antisymmetric:

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ even permutation} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index} \end{cases} \quad (ijk \text{ cyclic})$$

$$\vec{x} \times \vec{y} = x^i \hat{e}_i \times \hat{e}_j y^j = \epsilon_{ij}^k x^i y^j \hat{e}_k$$

~ orthogonal projection: $\hat{n} \times$ projects \perp to \hat{n} and rotates by 90°

$$\hat{x}_{\perp} = -\hat{n} \times (\hat{n} \times \hat{x}) = P_{\perp} \hat{x} \quad P_{\perp} = -\hat{n} \times \hat{n} \times$$

$$P_{\parallel} + P_{\perp} = \hat{n} \hat{n} \cdot -\hat{n} \times \hat{n} \times = I$$

~ where is the metric in \times ?

vector \times vector = pseudovector

symmetries act more like a 'bivector'

can be defined without metric

* triple product (volume of parallelepiped) - base times height $d = \vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

~ completely antisymmetric - definition of determinant

~ why is the scalar product symmetric / vector product antisymmetric?

~ vector vector \times vector = pseudoscalar (transformation properties)

~ acts more like a 'trivector' (volume element)

~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)

~ extended vector space with basis elements from objects of each dimension

~ pseudo-vectors, scalar separated from normal vectors, scalar

magnitude, length, area, volume

scalar, vectors, bivectors, trivector

$$1, \quad \hat{x}, \hat{y}, \hat{z}, \quad \hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}, \quad \hat{x}\hat{y}\hat{z}$$

~ what about higher-dimensional spaces (like space-time)?

can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements

most important example: BAC-CAB rule (HW: relation to projectors)

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$\epsilon_{ijk}^i A^j (\epsilon_{mn}^k B^m C^n) = (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A^j B^m C^n = B^i (A^j C_j) - C^i (A^j B_j)$$